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Kalman Filtering from the Perspective of the Heisenberg Uncertainty Principle

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Kalman Filtering from the Perspective of the Heisenberg Uncertainty Principle

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by

Yuqi Zhang

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Contents

List of Figures

Kalman Filtering from the Perspective of the Heisenberg Uncertainty Principle

Yuqi Zhang

1 Abstract

Finite escape means the occurrence of an infinite value in the solution of a time-varying differential equation. Finite escape can occur in the computation of either the Kalman or the Kalman-Bucy filter because the gain is time-varying. When no escape occurs it is analogous to the Heisenberg uncertainty principle [1] in atomic physics. Three noisy examples are given: a single integrator, a double integrator, and a linear oscillator. Finite escape cannot happen in the single integrator or the underdamped linear oscillator, but can happen in the double integrator and undamped linear oscillator. Therefore, finite escape can occur in the estimation of any noisy dynamic system. Except in special situations, it is impossible to achieve certainty in the determination of all state variables using neural nets, machine learning, or artificial intelligence even with an infinite amount of data. Conditions for finite escape to occur are given. Finally, practical solutions for escape are considered for the linear oscillator.

2 Acknowledgment

I would like to express my heartfelt gratitude to my advisor, Professor Donald Wiberg, for his invaluable guidance and support both academically and personally. His wisdom and meticulous mentorship have been pivotal to my growth and success. I am immensely thankful to my parents for their unconditional love and support. Their encouragement and understanding have enabled me to focus on pursuing my academic goals. Words cannot express how grateful I am for their unwavering support and love which I will forever cherish.

3 Introduction

Finite escape means the occurrence of an infinite value in the solution of a time-varying differential equation. Finite escape can occur in the computation of either the Kalman or the Kalman-Bucy filter because the gain is time-varying. Three simple examples of noisy linear systems are optimally filtered in continuous time by Kalman Bucy filtering, and the results are computed analytically. These examples can be generalized to higher-order systems that are time-varying and nonlinear. However, results for higher-order systems become too complicated to analyze analytically. The methods used for these simple examples can be extended to more complicated systems, making it feasible for linear cases that specify initial conditions and known inputs. These examples indicate a similarity to the Heisenberg's uncertainty principle extended to noisy linear dynamic systems. Furthermore, we show that the Kalman filter is not robust with respect to frequency variations in the linear oscillator, but it becomes robust when the unscented Kalman filter is used.

The three examples are: 1. a single integrator, 2. a double integrator, and 3. a linear oscillator.

Finite escape of the Kalman filter gain means that at some time $t^* < \infty$ the value of the gain becomes infinite. For practical applications, this is unacceptable. This finite escape occurs when the solution of the error variance Riccati equation, the $n \times n$ matrix $P(t)$, is no longer positive definite.

For applications, finite escape implies that the noisy linear model no longer reflects reality. A basic assumption used in the Kalman filtering model has been violated. Usually, the violated assumption is either an incorrect model order or departure from linearity. An interval of time surrounding the finite escape time must be avoided in practical applications. Infinite gain in feedback control causes instability and infinite control power demand.

A possible practical solution to finite escape has been proposed by [2]. Their solution involves controlled switches and both high and low pass filters. Possibly other solutions might also be appropriate.

4 Example 1: Single Integrator

In this example, the model is a single RC integrator. The system and observation model are defined in equation (1) and equation (2), $v(t)$ as the process noise and the $w(t)$ as the measurement noise. They conform to a Gaussian distribution. We choose the RC integrator to start from a one-dimensional system, then a double RC integrator to analyze the Kalman-Bucy Filter(KBF), and the Heisenberg Uncertainty Principle(HUP) in example 2.

Define x_0 as the Initial State. y_0 as the Initial Measurement.

$$\dot{x} = 0 + v(t)\ , x(0) = x_0\ , v \sim N(0, q^2) \tag{1}$$

$$y(t) = x(t) + w(t)\ , y(0) = y_0,\ w \sim N(0, r^2) \tag{2}$$

This is a noisy linear dynamical system. The evolution of this system over time is governed by a linear equation, but there is an additional component of randomness introduced by the noise. The process noise reflects incomplete modeling of the actual system. The measurement noise reflects imperfections in sensors or measurement devices, including random errors, quantization errors, and device noise.

The process to apply the Kalman-Bucy Filter is shown below, equation (3) to equation (9). In equation (3), K is the Kalman Gain, $\hat{x}(-)$ is the predicted estimate, $\hat{x}(+)$ is the updated estimate, and they are time-varying.

$$\hat{x}(+) = \hat{x}(-) + K(y - \hat{x}(-)) \tag{3}$$

$$P(+) = (I - K)P(-) \tag{4}$$

In equation (4), P is the error covariance, $P(+)$ is the posteriori error covariance, $P(-)$ is the priori error covariance, I is a $n \times n$ identity matrix, here $n = 1$. Now consider a discrete time interval equals δ.

Continuous prediction and update give [3]:

$$K(t) = P(t)R^{-1}(t) \tag{5}$$

$$x(t+\delta) - x(t) = x(t) + K(y(t) - x(t)) - x(t) = K(y(t) - x(t)) \tag{6}$$

$$P(t+\delta) - P(t) = P(+) - P(-) = (I - K)P(t) - P(t) = -P(t+\delta)R^{-1}P(t) \tag{7}$$

In equation (5), $R^{-1}(t)$ equals r^2. In equation (9), $Q(t)$ equals q^2.

Therefore, in the limit of $\delta \longrightarrow 0$

$$\dot{\hat{x}}(t) = K(y(t) - \hat{x}(t)) \tag{8}$$

$$\dot{P}(t) = Q(t) - P(t)R^{-1}P(t) \tag{9}$$

In the equation (1) and equation (2), v and w conform to a Gaussian probability distribution. The mean of v and w is 0, and the standard deviations are q and r.

$$\begin{cases} q & = 0.05\ , v \sim N(0, 0.0025) \\ r & = 0.1\ , w \sim N(0, 0.01) \end{cases} \tag{10}$$

In equation (11), $\hat{x}_0$ as the Initial State Estimate. P_0 as the Initial Estimation Error Covariance Matrix. Here P_0 is a constant.

$$\begin{cases} x_0 & = 0.5 \\ y_0 & = 0.54 \\ \hat{x}_0 & = 0.5 \\ P_0 & = \pi_0 = 1.2 \end{cases} \tag{11}$$

The MATLAB code for the Kalman-Bucy Filter is in Apppendices Listing 1.

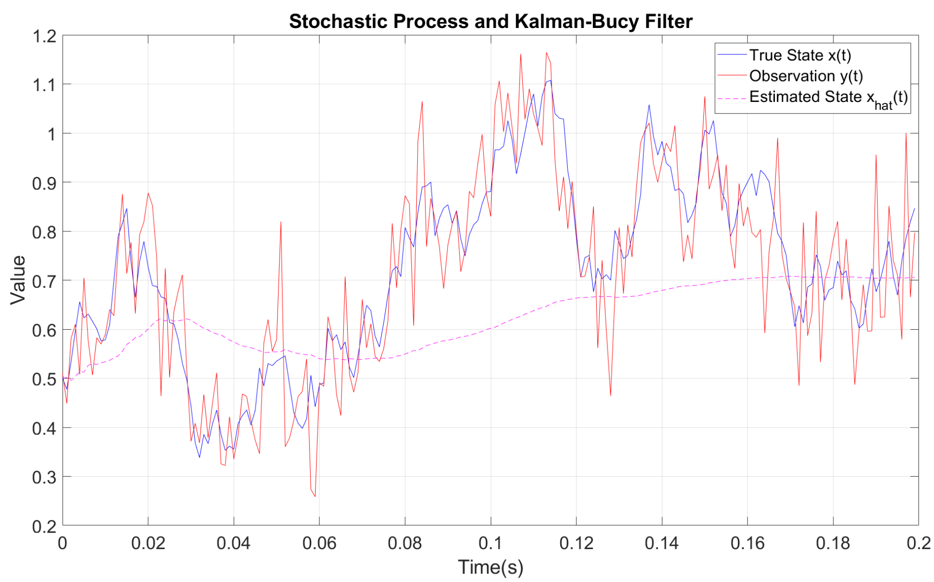

Fig. 1. Stochastic Process and Kalman-Bucy Filter for single integrator

In Figure 1. The blue trajectory is the random walk, according to the dynamic system model. The red trajectory is the measurement. The measurement model is a random walk with a Gaussian distributed noise. So the red line follows the blue line. The red dashed line is the output of the Kalman-Bucy Filter. The output is the optimal estimate of the state of the dynamic system.

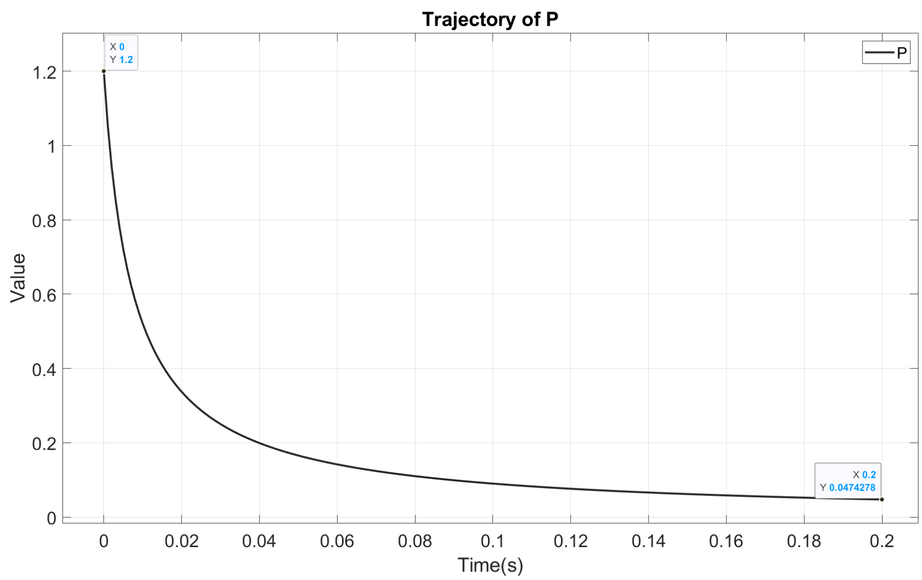

Fig. 2. Trajectory of P(t) for single integrator

In Figure 2. The trajectory of P starts from the initial guess P_0, which is 1.2. Then as the time goes on, the value of it keeps declining and tends to a constant. Its trajectory conforms to a hyperbolic function.

4.1 A Method to obtain $P(t)$

The mathematical method to obtain $P(t)$ is shown below.

$$P(t)X(t) = Y(t),\ P(t) = Y(t)X^{-1}(t),\ if\ det(X) \neq 0 \tag{12}$$

In equation (12), the relationship is between these three $n \times n$ matrix $P(t)$, $X(t)$ and $Y(t)$. $det(X)$ is the determinant of the $n \times n$ matrix X, here $n = 1$. Then

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = e^{Ht} \begin{pmatrix} 1 \\ \pi_0 \end{pmatrix}. \tag{13}$$

In equation(14), H is the Hamiltonian matrix defined in this 2×2 case as

$$H = \begin{bmatrix} 0 & \frac{1}{r^2} \\ q^2 & 0 \end{bmatrix}. \tag{14}$$

Denoting Laplace transformation as L, then inverse transformation gives

$$L^{-1}\left\{(sI - H)^{-1}\right\} = e^{Ht}. \tag{15}$$

In equation (15), L^{-1} is the inverse Laplace transformation. I is the identity matrix.

$$sI - H = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{r^2} \\ q^2 & 0 \end{pmatrix} = \begin{pmatrix} s & -\frac{1}{r^2} \\ -q^2 & s \end{pmatrix} \tag{16}$$

$$(sI - H)^{-1} = \frac{1}{s^2 - \frac{q^2}{r^2}} \begin{pmatrix} s & \frac{1}{r^2} \\ q^2 & s \end{pmatrix} = \begin{pmatrix} \frac{s}{s^2 - \frac{q^2}{r^2}} & \frac{r^{-2}}{s^2 - \frac{q^2}{r^2}} \\ \frac{q^2}{s^2 - \frac{q^2}{r^2}} & \frac{s}{s^2 - \frac{q^2}{r^2}} \end{pmatrix} \tag{17}$$

$$L^{-1}\left\{(sI - H)^{-1}\right\} = e^{Ht} = \begin{pmatrix} \cosh\frac{q}{r}t & \frac{1}{qr}\sinh\frac{q}{r}t \\ qr\sinh\frac{q}{r}t & \cosh\frac{q}{r}t \end{pmatrix} \tag{18}$$

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = e^{Ht} \begin{pmatrix} 1 \\ \pi_0 \end{pmatrix} = \begin{pmatrix} \cosh \frac{q}{r}t + \frac{\pi_0}{qr} \sinh \frac{q}{r}t \\ qr \sinh \frac{q}{r}t + \pi_0 \cosh \frac{q}{r}t \end{pmatrix} \tag{19}$$

$$P(t) = \frac{qr \sinh \frac{q}{r}t + \pi_0 \cosh \frac{q}{r}t}{\cosh \frac{q}{r}t + \frac{\pi_0}{qr} \sinh \frac{q}{r}t} = \frac{qr \tanh \frac{q}{r}t + \pi_0}{1 + \frac{\pi_0}{qr} \tanh \frac{q}{r}t} \tag{20}$$

When $t \to \infty$, $\tanh \frac{q}{r}t = 1$. Then

$$P(\infty) = \frac{qr + \pi_0}{1 + \frac{\pi_0}{qr}} = \frac{qr(qr + \pi_0)}{qr + \pi_0} = qr. \tag{21}$$

The MATLAB code for normalized $P(t)$ is in Appendices Listing 1.

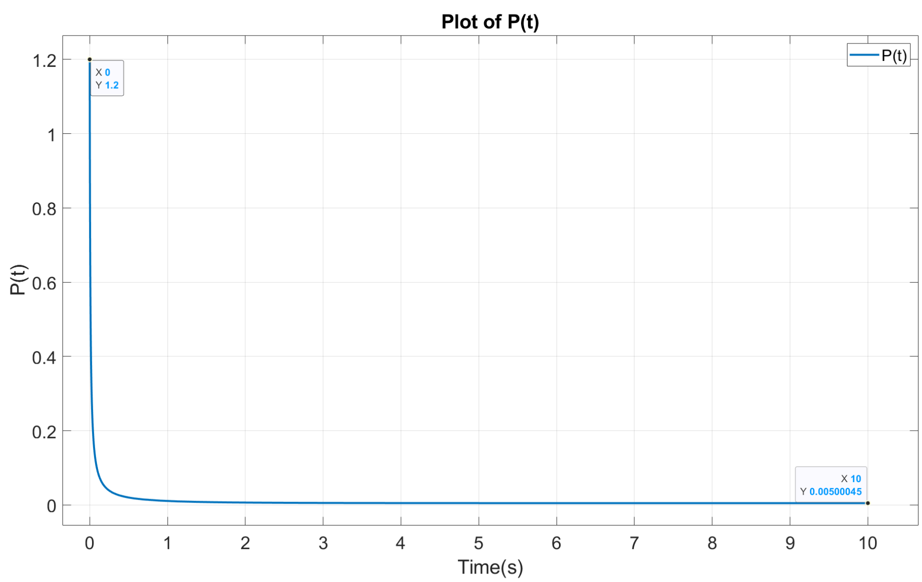

Fig. 3. Plot of P(t)

In Figure 3, the value of P starts from π_0, which is 1.2. As time increases, the value of P approaches qr, which is 0.005. The trajectory of P is a hyperbolic function, and the value of P is consistent with equation (21).

Zero measurement and process noise case:

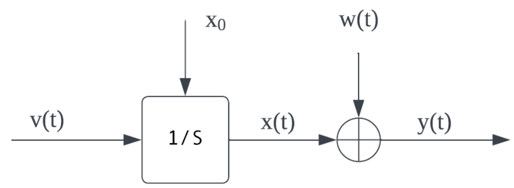

Fig. 4. Single noisy integrator

If $q = 0$, then $v(t) = 0$. Therefore, $\hat{x}(t)$ is determined by recursive least square estimation as

$$\hat{x}(t) = \hat{x}(0) + \frac{1}{t}\int_0^t y(\tau)\mathrm{d}\tau \to x_0. \tag{22}$$

If $r = 0$, then $w(t) = 0$. Therefore, using E as the conditional expectation operator,

$$\hat{x}(t) = \frac{1}{t}E\left\{\int_0^t y(\tau)\mathrm{d}\tau\right\} = \frac{1}{t}E\left\{\int_0^t \left[x_0 + v(\tau)\right]\mathrm{d}\tau\right\} \to x_0 \tag{23}$$

Since there is no measurement noise, $x(t) = \hat{x}(t)$ after $t = 0$.

From the single integrator of example 1, the only occurrence of finite escape is when there is no measurement noise($r = 0$). When an 'infinite' amount of data has been collected as $t \to \infty$, for $q = 0$, then the least square estimate yields $\hat{x}(t) = x_0 = x(t)$ with probability one ('certainty'). Otherwise, when neither q nor r is zero, the error variance of the estimate $\hat{x}(t)$ tends to $P(\infty)$ as $t = \infty$, where $P(\infty) = qr > 0$. Therefore,

$\hat{x}$ cannot ever be determined exactly even with an infinite amount of data as $t \to \infty$. This is the effect of unknown 'infinite condition', unknown measurement noise, and unknown input noise.

5 Example 2: Double Integrator

The double integrator is the case of two single integrators in series connection.

System equations:

$$\begin{cases} \dot{x_1} = x_2,\ x_1(0) = x_{1_0} \\ \dot{x_2} = 0,\ x_2(0) = x_{2_0} \end{cases} \tag{24}$$

Measurement:

$$y(t) = x_1(t) \tag{25}$$

In equations (24) and (25), define a continuous time second-order dynamical system, which is very common. For example, $x_1(t)$ is the location of an object and $x_2(t)$ is the acceleration of the object.

Model with noise:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v \tag{26}$$

Measurement:

$$y(t) = x_1(t) + w \tag{27}$$

v and w are scalar valued white noise :

$$\mathrm{d}v \sim N(0, q^2\mathrm{d}t),\ q = 0.5 \tag{28}$$

$$\mathrm{d}w \sim N(0, r^2\mathrm{d}t),\ r = 0.5 \tag{29}$$

In equation (26) and equation (27), the process noise and the measurement noise are added. Then, v and ω conform to a Gaussian distribution, the mean of v and w is 0, and the standard deviations are q and r.

Equations for the Kalman-Bucy Filter are shown in equation (30) through equation [3].

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + K(t)(y(t) - H(t)\hat{x}(t)) \tag{30}$$

$$K(t) = \begin{pmatrix} K_1(t) \\ K_2(t) \end{pmatrix} = \begin{pmatrix} \frac{P_{11}(t)}{r^2} \\ \frac{P_{12}(t)}{r^2} \end{pmatrix} \tag{31}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \hat{x_1} \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} \hat{x_2}(t) \\ 0 \end{pmatrix} + \begin{pmatrix} K_1(t)(y(t) - \hat{x_1}(t)) \\ K_2(t)(y(t) - \hat{x_1}(t)) \end{pmatrix} \tag{32}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = AP(t) + P(t)A^T + GQG^t - P(t)H^T R^{-1} HP(t) \tag{33}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \begin{pmatrix} 2P_{12}(t) - P_{11}^2(t)/r^2 & P_{22}(t) - P_{11}(t)P_{12}(t)/r^2 \\ P_{22}(t) - P_{12}(t)P_{11}(t)/r^2 & q^2 - P_{12}^2(t)/r^2 \end{pmatrix} \tag{34}$$

Initial Conditions:

$$\begin{cases} x_1(0) & = 1 \ , \hat{x_1}(t) = 1.4 \\ x_2(0) & = 1 \ , \hat{x_2}(t) = 0.6 \\ y_1(0) & = 1.04 \end{cases} \begin{cases} P_{11}(0) & = \pi_1 \ , \pi_1 = 1 \\ P_{22}(0) & = \pi_2 \ , \pi_2 = 1 \\ P_{12}(0) & = P_{21}(0) = 0 \end{cases} \tag{35}$$

The MATLAB code for the Kalman-Bucy Filter is in Appendices Listing 3.

The Output of the Kalman-Bucy Filter:

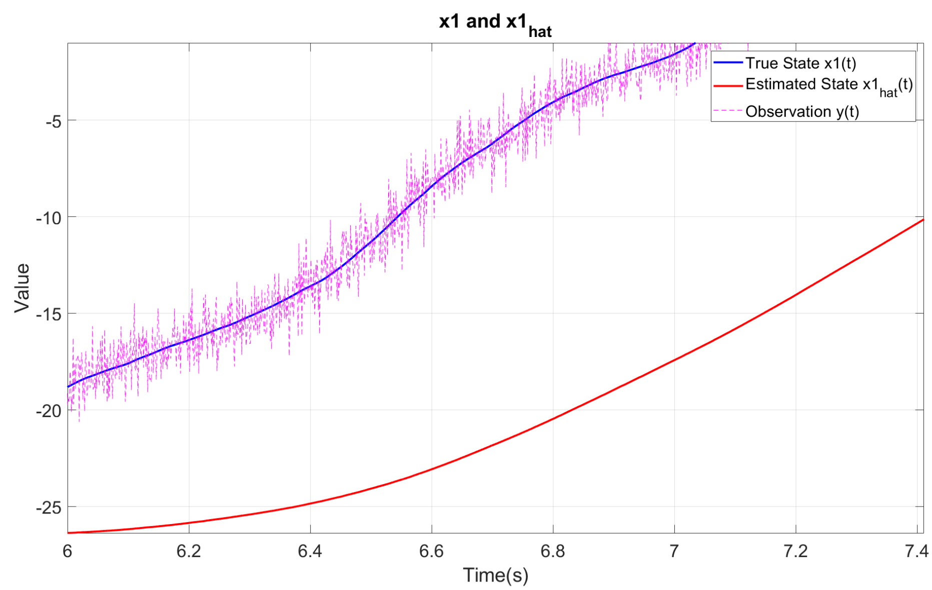

Fig. 5. Double Integrator Model output and its $x_1(t)$ *and* $\hat{x}_1(t)$ plotted.

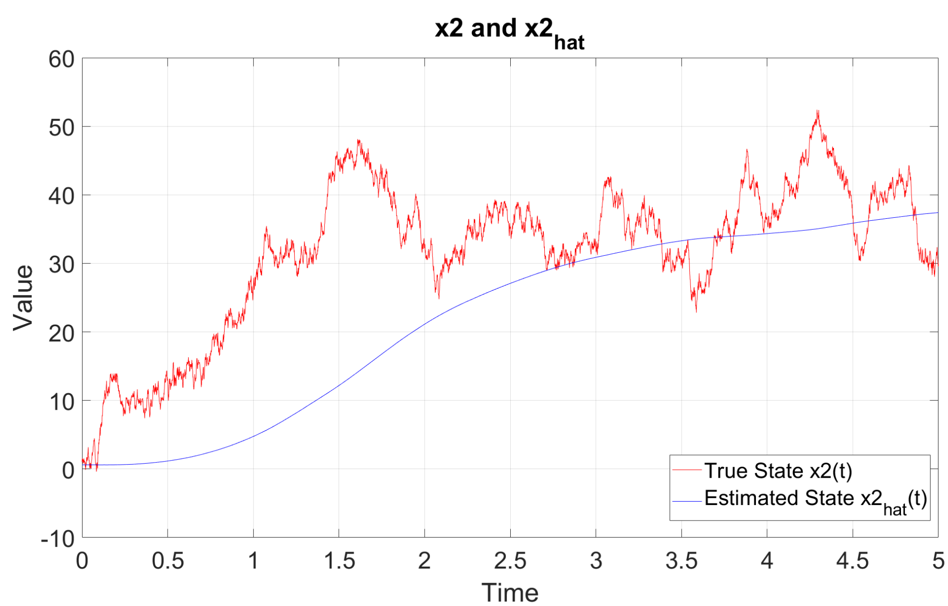

Fig. 6. Double Integrator unobserved state and its estimate $x_2(t)$ *and* $\hat{x}_2(t)$ plotted.

In Figure 5, the blue trajectory is $x_1(t)$ which is a cumulative value. The red dashed line is the trajectory of the measurement. The measurement model is a cumulative value with a Gaussian distributed noise. So the red dashed follows the blue line. The red one is the output of the Kalman-Bucy Filter. The output is the optimal estimate of the states of the $x_1(t)$. In Figure 6, the red trajectory is $x_2(t)$, a random walk, agreeing with the dynamic system model. The blue trajectory is $\hat{x}_2(t)$. Notice that the Kalman-Bucy Filter eliminated the white noise, defined by v.

5.1 A Method to obtain $P(t)$

Above, P, X, and Y have been defined. Here, $n = 2$.

$$P(t)X(t) = Y(t),\ P(t) = Y(t)X^{-1}(t),\ if\ det(X) \neq 0 \tag{36}$$

In equation (36), the relationship between $P(t)$, $X(t)$ and $Y(t)$ is given. $P(t)$ is the error covariance matrix, $X(t)$ is the state matrix and $Y(t)$ is the measurement matrix. $det(X)$ is the determinate of $X(t)$.

So,

$$\frac{\mathrm{d}(PX)}{\mathrm{d}t} = X\frac{\mathrm{d}P}{\mathrm{d}t} + P\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{\mathrm{d}Y}{\mathrm{d}t} \tag{37}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X \\ Y \end{pmatrix} = H\begin{pmatrix} X \\ Y \end{pmatrix}. \tag{38}$$

In equation (38), H is the Hamiltonian matrix. Note X and Y are 2×2 matrices. To obtain stable versions of the 2×2 X and Y matrices, both must be divided by the largest time function, which in this case is $e^{\omega t}$. In equation (39), h^T is the 2-row vector transforming the model state x to the measurement of y with the condition of the noise w. A, Q, R, G, and h are defined in Grewal & Andrews [3].

$$H = \begin{bmatrix} -A^T & h^T R^{-1} h \\ GQG^T & A \end{bmatrix} \tag{39}$$

$$H = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & q^2 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \tag{40}$$

$$\frac{\mathrm{d}X}{\mathrm{d}Y} = \begin{pmatrix} -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & r^{-2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ q^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \tag{41}$$

$$X(0) = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \, Y(0) = P_0 = \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix} \pi_1 > 0, \pi_2 > 0 \tag{42}$$

Using the Laplace transform function

$$s \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} - H \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} X(0) \\ Y(0) \end{pmatrix} \tag{43}$$

$$e^{Ht} = L^{-1}\{(sI-H)^{-1}\} \tag{44}$$

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = e^{Ht} \begin{pmatrix} X(0) \\ Y(0) \end{pmatrix}. \tag{45}$$

From Leverrier's algorithm [4],

$$(sI_4 - A)^{-1} = \frac{s^3F_1 + s^2F_2 + sF_3 + F_4}{s^4 + \theta_1 s^3 + \theta_2 s^2 + \theta_3 s + \theta_4} \tag{46}$$

F_1,F_2,F_3,$F_4$are 4×4 real matrices, θ_1, θ_2, θ_3, θ_4 are real scalars

$$\begin{cases} F_1 & = I_4 \\ F_2 & = HF_1 + \theta_1 I_4 \\ F_3 & = HF_2 + \theta_2 I_4 \\ F_4 & = HF_3 + \theta_3 I_4, \end{cases} \begin{cases} \theta_1 & = -trHF_1/1 \\ \theta_2 & = -trHF_2/2 \\ \theta_3 & = -trHF_3/3 \\ \theta_4 & = -trHF_4/4 \end{cases} \tag{47}$$

$$(sI_4 - A)^{-1} = \frac{\begin{bmatrix} s^3 & \frac{q^2}{r^2} & \frac{s^2}{r^2} & \frac{s}{r^2} \\ -s^2 & s^3 & -\frac{s}{r^2} & -\frac{1}{r^2} \\ -q^2 & q^2 s & s^3 & s^2 \\ -q^2 s & q^2 s^2 & -\frac{q^2}{r^2} & s^3 \end{bmatrix}}{s^4 + 0\times s^3 + 0\times s^2 + 0\times s + \frac{q^2}{r^2}} = \frac{\begin{bmatrix} s^3 & \frac{q^2}{r^2} & \frac{s^2}{r^2} & \frac{s}{r^2} \\ -s^2 & s^3 & -\frac{s}{r^2} & -\frac{1}{r^2} \\ -q^2 & q^2 s & s^3 & s^2 \\ -q^2 s & q^2 s^2 & -\frac{q^2}{r^2} & s^3 \end{bmatrix}}{s^4 + \frac{q^2}{r^2}} \tag{48}$$

The eigenvalues of H are four values of λ, namely λ_1, λ_2, λ_3, λ_4:

$$\lambda = \left[\frac{-1+j}{\sqrt{2}}\sqrt{\frac{q}{r}}, \frac{-1-j}{\sqrt{2}}\sqrt{\frac{q}{r}}, \frac{1+j}{\sqrt{2}}\sqrt{\frac{q}{r}}, \frac{1-j}{\sqrt{2}}\sqrt{\frac{q}{r}}\right] \tag{49}$$

Using the eigenvalues of H, obtain x_i and their reciprocals, $r_i^\dagger$, as complex 4-vectors:

$$e^{Ht} = \sum_{i=1}^{n=4} e^{\lambda_i t} x_i r_i^\dagger \tag{50}$$

The process to calculate e^{Ht} is shown below.

$$x_1 = \begin{bmatrix} \frac{1-j}{\sqrt{2}}\sqrt{\frac{q}{r}} \\ 1 \\ jqr \\ \frac{-1-j}{\sqrt{2}}q\sqrt{qr} \end{bmatrix},\ x_2 = \begin{bmatrix} \frac{1+j}{\sqrt{2}}\sqrt{\frac{q}{r}} \\ 1 \\ -jqr \\ \frac{-1+j}{\sqrt{2}}q\sqrt{qr} \end{bmatrix},\ x_3 = \begin{bmatrix} \frac{-1-j}{\sqrt{2}}\sqrt{\frac{q}{r}} \\ 1 \\ -jqr \\ \frac{1-j}{\sqrt{2}}q\sqrt{qr} \end{bmatrix},\ x_4 = \begin{bmatrix} \frac{-1+j}{\sqrt{2}}\sqrt{\frac{q}{r}} \\ 1 \\ jqr \\ \frac{1+j}{\sqrt{2}}q\sqrt{qr} \end{bmatrix} \tag{51}$$

$$(x_1|x_2|x_3|x_4)^{-1} = \begin{bmatrix} r_1^\dagger \\ r_2^\dagger \\ r_3^\dagger \\ r_4^\dagger \end{bmatrix} \tag{52}$$

$$(x_1|x_2|x_3|x_4)^{-1} = \begin{bmatrix} \frac{1+j}{8}\sqrt{\frac{2r}{q}} & \frac{1}{4} & -\frac{j}{4qr} & \sqrt{\frac{2}{qr}}\frac{(-1+j)}{8q} \\ \frac{1-j}{8}\sqrt{\frac{2r}{q}} & \frac{1}{4} & \frac{j}{4qr} & \sqrt{\frac{2}{qr}}\frac{(-1-j)}{8q} \\ \frac{-1+j}{8}\sqrt{\frac{2r}{q}} & \frac{1}{4} & \frac{j}{4qr} & \sqrt{\frac{2}{qr}}\frac{(1+j)}{8q} \\ \frac{-1-j}{8}\sqrt{\frac{2r}{q}} & \frac{1}{4} & -\frac{j}{4qr} & \sqrt{\frac{2}{qr}}\frac{(1-j)}{8q} \end{bmatrix} \tag{53}$$

$$r_1^\dagger = \begin{bmatrix} \frac{1+j}{8}\sqrt{\frac{2r}{q}} & \frac{1}{4} & -\frac{j}{4qr} & \sqrt{\frac{2}{qr}}\frac{(-1+j)}{8q} \end{bmatrix} \tag{54}$$

$$r_2^\dagger = \begin{bmatrix} \frac{1-j}{8}\sqrt{\frac{2r}{q}} & \frac{1}{4} & \frac{j}{4qr} & \sqrt{\frac{2}{qr}}\frac{(-1-j)}{8q} \end{bmatrix} \tag{55}$$

$$r_3^\dagger = \begin{bmatrix} \frac{-1+j}{8}\sqrt{\frac{2r}{q}} & \frac{1}{4} & \frac{j}{4qr} & \sqrt{\frac{2}{qr}}\frac{(1+j)}{8q} \end{bmatrix} \tag{56}$$

$$r_4^\dagger = \begin{bmatrix} \frac{-1-j}{8}\sqrt{\frac{2r}{q}} & \frac{1}{4} & -\frac{j}{4qr} & \sqrt{\frac{2}{qr}}\frac{(1-j)}{8q} \end{bmatrix} \tag{57}$$

$$x_1 r_1^\dagger = \begin{bmatrix} \frac{1}{4} & \sqrt{\frac{2q}{r}}\frac{(1-j)}{8} & \sqrt{\frac{2}{qr}}\frac{(-1-j)}{8r} & \frac{j}{4qr} \\ \frac{1+j}{8}\sqrt{\frac{2r}{q}} & \frac{1}{4} & -\frac{j}{4qr} & \sqrt{\frac{2}{qr}}\frac{(-1+j)}{8q} \\ r\sqrt{2qr}\frac{(-1+j)}{8} & \frac{qrj}{4} & \frac{1}{4} & \sqrt{\frac{2r}{q}}\frac{(-1-j)}{8} \\ -\frac{qrj}{4} & q\sqrt{2qr}\frac{(-1-j)}{8} & \sqrt{2qr}\frac{(-1+j)}{8r} & \frac{1}{4} \end{bmatrix} \tag{58}$$

$$x_2 r_2^\dagger = \begin{bmatrix} \frac{1}{4} & \sqrt{\frac{2q}{r}}\frac{(1+j)}{8} & \sqrt{\frac{2}{qr}}\frac{(-1+j)}{8r} & -\frac{j}{4qr} \\ \frac{1-j}{8}\sqrt{\frac{2r}{q}} & \frac{1}{4} & \frac{j}{4qr} & \sqrt{\frac{2}{qr}}\frac{(-1-j)}{8q} \\ r\sqrt{2qr}\frac{(-1-j)}{8} & -\frac{qrj}{4} & \frac{1}{4} & \sqrt{\frac{2r}{q}}\frac{(-1+j)}{8} \\ \frac{qrj}{4} & q\sqrt{2qr}\frac{(-1+j)}{8} & \sqrt{2qr}\frac{(-1-j)}{8r} & \frac{1}{4} \end{bmatrix} \tag{59}$$

$$x_3 r_3^\dagger = \begin{bmatrix} \frac{1}{4} & \sqrt{\frac{2q}{r}}\frac{(-1-j)}{8} & \sqrt{\frac{2}{qr}}\frac{(1-j)}{8r} & -\frac{j}{4qr} \\ \frac{-1+j}{8}\sqrt{\frac{2r}{q}} & \frac{1}{4} & \frac{j}{4qr} & \sqrt{\frac{2}{qr}}\frac{(1+j)}{8q} \\ r\sqrt{2qr}\frac{(1+j)}{8} & -\frac{qrj}{4} & \frac{1}{4} & \sqrt{\frac{2r}{q}}\frac{(1-j)}{8} \\ \frac{qrj}{4} & q\sqrt{2qr}\frac{(1-j)}{8} & \sqrt{2qr}\frac{(1+j)}{8r} & \frac{1}{4} \end{bmatrix} \tag{60}$$

$$x_4 r_4^\dagger = \begin{bmatrix} \frac{1}{4} & \sqrt{\frac{2q}{r}}\frac{(-1+j)}{8} & \sqrt{\frac{2}{qr}}\frac{(1+j)}{8r} & \frac{j}{4qr} \\ \frac{-1-j}{8}\sqrt{\frac{2r}{q}} & \frac{1}{4} & -\frac{j}{4qr} & \sqrt{\frac{2}{qr}}\frac{(1-j)}{8q} \\ r\sqrt{2qr}\frac{(1-j)}{8} & \frac{qrj}{4} & \frac{1}{4} & \sqrt{\frac{2r}{q}}\frac{(1+j)}{8} \\ -\frac{qrj}{4} & q\sqrt{2qr}\frac{(1+j)}{8} & \sqrt{2qr}\frac{(1-j)}{8r} & \frac{1}{4} \end{bmatrix} \tag{61}$$

Applying equation (50), the expression for e^{Ht} becomes

$$e^{Ht} = e^{\lambda_1 t} x_1 r_1^\dagger + e^{\lambda_2 t} x_2 r_2^\dagger + e^{\lambda_3 t} x_3 r_3^\dagger + e^{\lambda_4 t} x_4 r_4^\dagger \tag{62}$$

$$= e^{(-1+j)\sqrt{\frac{q}{2r}}t} x_1 r_1^\dagger + e^{(-1-j)\sqrt{\frac{q}{2r}}t} x_2 r_2^\dagger + e^{(1+j)\sqrt{\frac{q}{2r}}t} x_3 r_3^\dagger + e^{(1-j)\sqrt{\frac{q}{2r}}t} x_4 r_4^\dagger. \tag{63}$$

Define $\omega = \sqrt{\frac{q}{2r}}$ and obtain

$$e^{Ht} = e^{-\omega t}\left(e^{jwt} x_1 r_1^\dagger + e^{-jwt} x_2 r_2^\dagger\right) + e^{\omega t}\left(e^{jwt} x_3 r_3^\dagger + e^{-jwt} x_4 r_4^\dagger\right). \tag{64}$$

Equation (64) consists of two parts. $e^{-\omega t}\left(e^{jwt} x_1 r_1^\dagger + e^{-jwt} x_2 r_2^\dagger\right)$ is the first part.
$e^{\omega t}\left(e^{jwt} x_3 r_3^\dagger + e^{-jwt} x_4 r_4^\dagger\right)$ is the second part.
The expression of $\left(e^{jwt} x_1 r_1^\dagger + e^{-jwt} x_2 r_2^\dagger\right)$ is shown as below, a 4×4 matrix.
First row:

$$\begin{cases} (1,1)\ \frac{1}{2}\cos\omega t \\ (1,2)\ \frac{1}{4}\sqrt{\frac{2q}{r}}(\cos\omega t + \sin\omega t) \\ (1,3)\ -\frac{1}{4r}\sqrt{\frac{2}{qr}}(\cos\omega t - \sin\omega t) \\ (1,4)\ -\frac{1}{2qr}\sin\omega t \end{cases} \tag{65}$$

Second row:

$$\begin{cases} (2,1)\ \frac{1}{4}\sqrt{\frac{2r}{q}}(\cos\omega t - \sin\omega t) \\ (2,2)\ \frac{1}{2}\cos\omega t \\ (2,3)\ \frac{1}{2qr}\sin\omega t \\ (2,4)\ -\frac{1}{4q}\sqrt{\frac{2}{qr}}(\cos\omega t + \sin\omega t) \end{cases} \tag{66}$$

Third row:

$$\begin{cases} (3,1)\ -\frac{r\sqrt{2qr}}{4}(\cos\omega t + \sin\omega t) \\ (3,2)\ -\frac{qr}{2}\sin\omega t \\ (3,3)\ \frac{1}{2}\cos\omega t \\ (3,4)\ -\frac{1}{4}\sqrt{\frac{2r}{q}}(\cos\omega t - \sin\omega t) \end{cases} \tag{67}$$

Fourth row:

$$\begin{cases} (4,1)\ \frac{qr}{2}\sin\omega t \\ (4,2)\ -\frac{q\sqrt{2qr}}{4}(\cos\omega t - \sin\omega t) \\ (4,3)\ -\frac{\sqrt{2qr}}{4r}(\cos\omega t + \sin\omega t) \\ (4,4)\ \frac{1}{2}\cos\omega t \end{cases} \tag{68}$$

The expression of $\left(e^{jwt}x_3r_3^\dagger + e^{-jwt}x_4r_4^\dagger\right)$ is shown below, a 4×4 matrix.

First row:

$$\begin{cases} (1,1)\ \frac{1}{2}\cos\omega t \\ (1,2)\ -\frac{1}{4}\sqrt{\frac{2q}{r}}(\cos\omega t - \sin\omega t) \\ (1,3)\ \frac{1}{4r}\sqrt{\frac{2}{qr}}(\cos\omega t + \sin\omega t) \\ (1,4)\ \frac{1}{2qr}\sin\omega t \end{cases} \tag{69}$$

Second row:

$$\begin{cases} (2,1)\ -\frac{1}{4}\sqrt{\frac{2r}{q}}(\cos\omega t + \sin\omega t) \\ (2,2)\ \frac{1}{2}\cos\omega t \\ (2,3)\ -\frac{1}{2qr}\sin\omega t \\ (2,4)\ \frac{1}{4q}\sqrt{\frac{2}{qr}}(\cos\omega t - \sin\omega t) \end{cases} \tag{70}$$

Third row:

$$\begin{cases} (3,1)\ \frac{r\sqrt{2qr}}{4}(\cos\omega t - \sin\omega t) \\ (3,2)\ -\frac{qr}{2}\sin\omega t \\ (3,3)\ \frac{1}{2}\cos\omega t \\ (3,4)\ \frac{1}{4}\sqrt{\frac{2r}{q}}(\cos\omega t + \sin\omega t) \end{cases} \tag{71}$$

Fourth row:

$$\begin{cases} (4,1) \ -\frac{qr}{2}\sin\omega t \\ (4,2) \ \frac{q\sqrt{2qr}}{4}(\cos\omega t+\sin\omega t) \\ (4,3) \ \frac{\sqrt{2qr}}{4r}(\cos\omega t-\sin\omega t) \\ (4,4) \ \frac{1}{2}\cos\omega t \end{cases} \tag{72}$$

The expression of e^{Ht} are shown below, a 4×4 matrix.

First row:

$$\begin{cases} (1,1) \ \cos\omega t\cosh\omega t \\ (1,2) \ -\sqrt{\frac{q}{2r}}(\cos\omega t\sinh\omega t-\sin\omega t\cosh\omega t) \\ (1,3) \ \frac{1}{r\sqrt{2qr}}(\cos\omega t\sinh\omega t+\sin\omega t\cosh\omega t) \\ (1,4) \ \frac{1}{qr}\sin\omega t\sinh\omega t \end{cases} \tag{73}$$

Second row:

$$\begin{cases} (2,1) \ -\sqrt{\frac{r}{2q}}(\cos\omega t\sinh\omega t+\sin\omega t\cosh\omega t) \\ (2,2) \ \cos\omega t\cosh\omega t \\ (2,3) \ -\frac{1}{qr}\sin\omega t\sinh\omega t \\ (2,4) \ \frac{1}{q\sqrt{2qr}}(\cos\omega t\sinh\omega t-\sin\omega t\cosh\omega t) \end{cases} \tag{74}$$

Third row:

$$\begin{cases} (3,1)\ \frac{r\sqrt{2qr}}{2}(\cos\omega t\sinh\omega t-\sin\omega t\cosh\omega t) \\ (3,2)\ qr\sin\omega t\sinh\omega t \\ (3,3)\ \cos\omega t\cosh\omega t \\ (3,4)\ \sqrt{\frac{r}{2q}}(\cos\omega t\sinh\omega t+\sin\omega t\cosh\omega t) \end{cases} \tag{75}$$

Fourth row:

$$\begin{cases} (4,1)\ -qr\sin\sinh \\ (4,2)\ \frac{q\sqrt{2qr}}{2}(\cos\omega t\sinh\omega t+\sin\omega t\cosh\omega t) \\ (4,3)\ \frac{\sqrt{2qr}}{2r}(\cos\omega t\sinh\omega t-\sin\omega t\cosh\omega t) \\ (4,4)\ \cos\omega t\cosh\omega t \end{cases} \tag{76}$$

After obtaining the expression of e^{Ht}, by using equation(41) the expression of $X(t)$ and $Y(t)$ becomes a 2×2 matrix. So the expression of $P(t)$ is obtained from the Kalman-Bucy Filter. When $q=0.5$ and $r=0.5$, then the trajectory of $P(t)$ can be plotted.

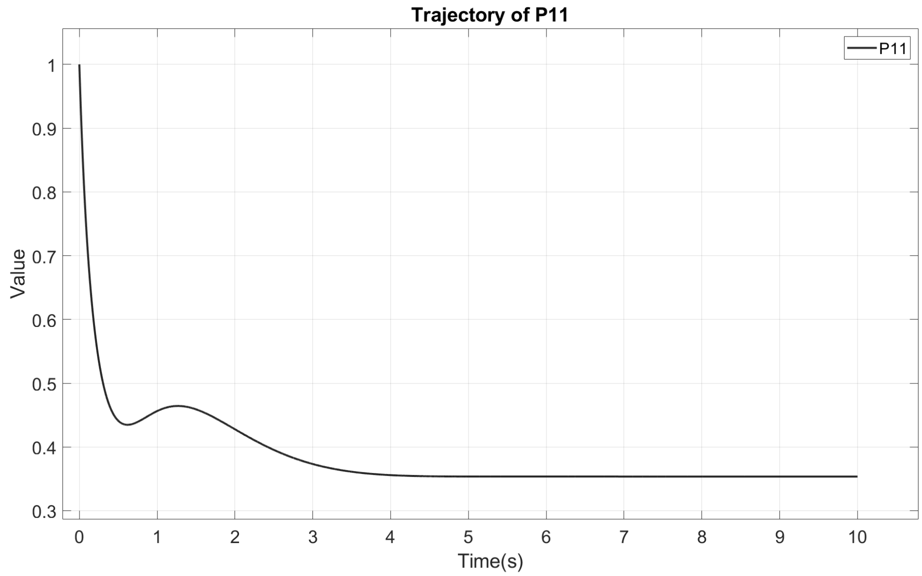

Fig. 7. Trajectory of $P11(t)$ for the double integrators

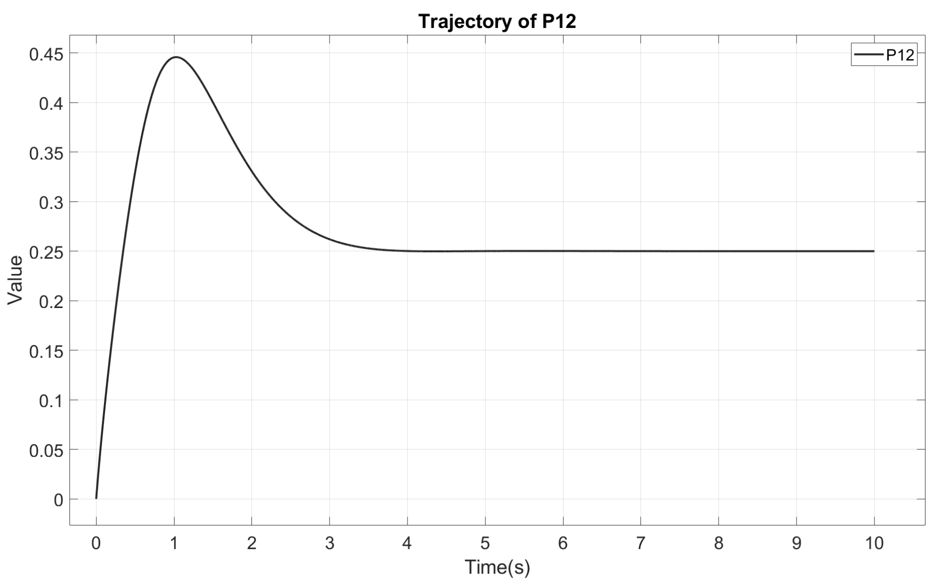

Fig. 8. Trajectory of $P12(t)$ for the double integrators

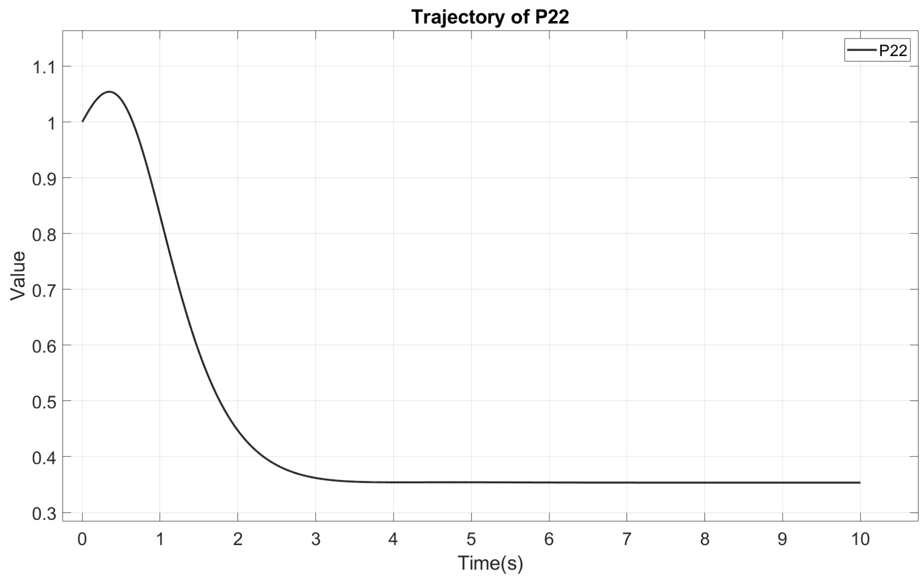

Fig. 9. Trajectory of $P22(t)$ for the double integrators

Figure 7 shows that $P_{11}(t)$ starts from $P_{11}(0)$ which is 1. Then the value declines very fast. At 0.5s, it tends to a steady state. Then it continues to slowly decrease at a reduced rate. Until to 2s, it becomes relatively flat. The Figure 8 shows that the $P_{12}(t)$ is starts from the $P_{12}(0)$ which is 0. The value of P12 rises fast at first. At 1s, the value reaches a local maximum, then keeps declining and tends to a constant. In Figure 9, $P_{22}(t)$ starts from $P_{22}(0)$ which is 1. The value of P22 rises first, then keeps declining and tends to a constant as the time increases.

$P11(t)$, $P12(t)$ and $P22(t)$ divided by $\cosh \omega t$ are shown below:

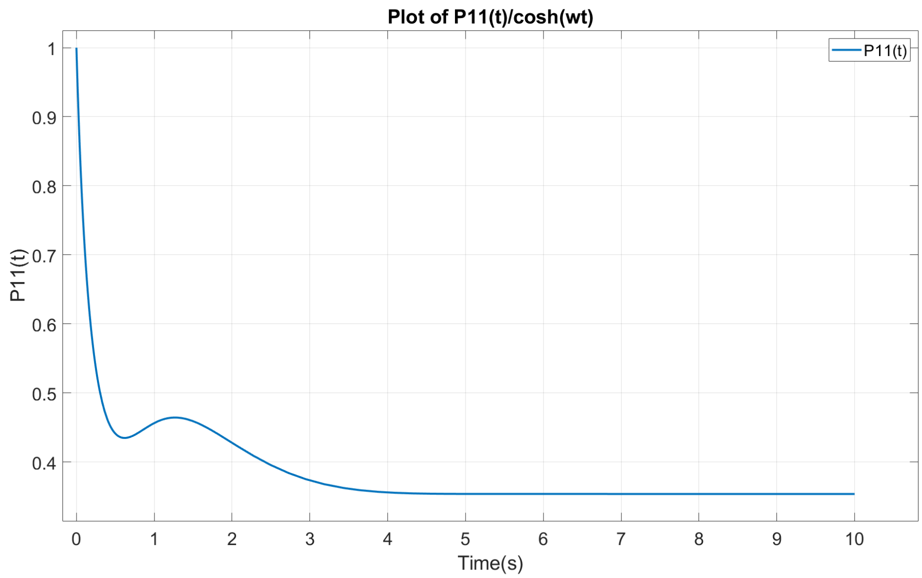

Fig. 10. Plot of $P11(t)/\cosh\omega t$

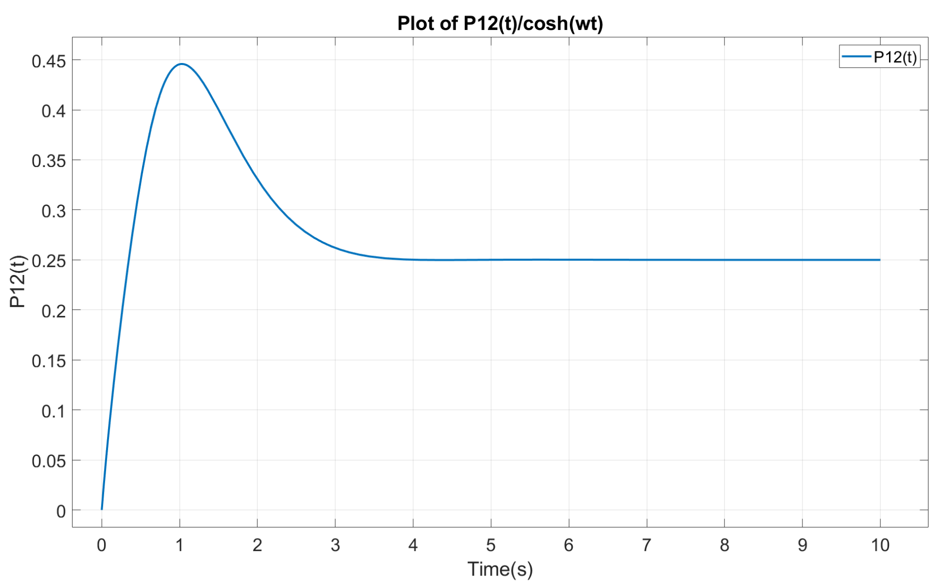

Fig. 11. Plot of $P12(t)/\cosh\omega t$

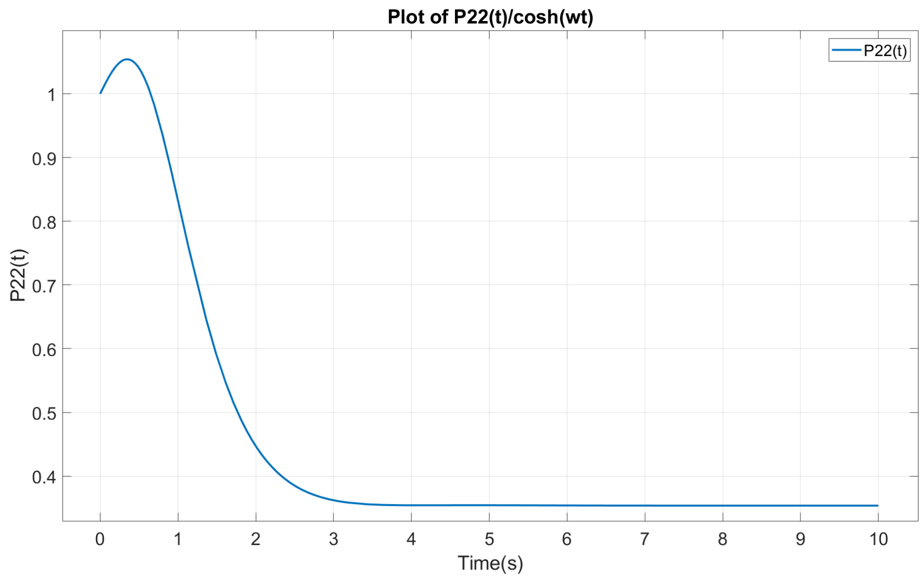

Fig. 12. Plot of $P22(t)/\cosh\omega t$

Compared with Figure 7, Figure 8 and Figure 9, the trend of the output of the Kalman-Bucy Filter is same with them. The mathematical expression of $P_{11}(\infty)$, $P_{12}(\infty)$ and $P_{22}(\infty)$ are shown below.

$$\begin{cases} \Lambda_1 = \left[\cos\omega t + \frac{\pi_1}{r\sqrt{2qr}}(\cos\omega t + \sin\omega t)\right]\left[\cos\omega t + \frac{\pi_2}{q\sqrt{2qr}}(\cos\omega t - \sin\omega t)\right] \\ \Lambda_2 = \left[-\sqrt{\frac{q}{2r}}(\cos\omega t - \sin\omega t) + \frac{\pi_2}{qr}\sin\omega t\right]\left[-\sqrt{\frac{r}{2q}}(\cos\omega t + \sin\omega t) - \frac{\pi_1}{qr}\sin\omega t\right] \end{cases} \tag{77}$$

$P_{11}(\infty)$:

$$\begin{cases} \Theta_1 &= \left[\frac{r\sqrt{2qr}}{2}(\cos\omega t - \sin\omega t) + \pi_1\cos\omega t\right]\left[\frac{\pi_2}{q\sqrt{2qr}}(\cos\omega t - \sin\omega t) + \cos\omega t\right] \\ \Theta_2 &= \left[qr\sin\omega t + \pi_2\sqrt{\frac{r}{2q}}(\cos\omega t + \sin\omega t)\right]\left[\sqrt{\frac{r}{2q}}(\cos\omega t + \sin\omega t) + \frac{\pi_1}{qr}\sin\omega t\right] \end{cases} \tag{78}$$

$$P_{11}(\infty) = \frac{\Theta_1 + \Theta_2}{\Lambda_1 - \Lambda_2} \tag{79}$$

$P_{12}(\infty)$:

$$\begin{cases} \Theta_3 &= \left[\frac{r\sqrt{2qr}}{2}(\cos\omega t - \sin\omega t) + \pi_1 \cos\omega t\right]\left[\sqrt{\frac{q}{2r}}(\cos\omega t - \sin\omega t) - \frac{\pi_2}{qr}\sin\omega t\right] \\ \Theta_4 &= \left[qr\sin\omega t + \pi_2\sqrt{\frac{r}{2q}}(\cos\omega t + \sin\omega t)\right]\left[\cos\omega t + \frac{\pi_1}{r\sqrt{2qr}}(\cos\omega t + \sin\omega t)\right] \end{cases} \tag{80}$$

$$P_{12}(\infty) = \frac{\Theta_3 + \Theta_4}{\Lambda_1 - \Lambda_2} \tag{81}$$

$P_{22}(\infty)$:

$$\begin{cases} \Theta_5 &= \left[-qr\sin\omega t + \frac{\pi_1\sqrt{2qr}}{2r}(\cos\omega t - \sin\omega t)\right]\left[\sqrt{\frac{q}{2r}}(\cos\omega t - \sin\omega t) - \frac{\pi_2}{qr}\sin\omega t\right] \\ \Theta_6 &= \left[\frac{q\sqrt{2qr}}{2}(\cos\omega t + \sin\omega t) + \pi_2 \cos\omega t\right]\left[\cos\omega t + \frac{\pi_1}{r\sqrt{2qr}}(\cos\omega t + \sin\omega t)\right] \end{cases} \tag{82}$$

$$P_{22}(\infty) = \frac{\Theta_5 + \Theta_6}{\Lambda_1 - \Lambda_2} \tag{83}$$

From equation (79), equation (81) and equation (83), the P matrix doesn't have a steady state as in the single integrator.

Considering again equation (36), there is a time t^* at which the determinant of $X(t)$ might equal to zero.

$$P(t)X(t) = Y(t) \tag{84}$$

$$det(P(t^*))det(X(t^*)) = det(Y(t^*)) \tag{85}$$

In equation (85), $det(P(t^*))$ is the determinant of P matrix at time t^*, $det(X(t^*))$ is the determinant of X matrix at time t^* and $det(Y(t^*))$ is the determinant of Y matrix at time t^*. The $Y(t)$ matrix starts at $Y(0) = P_0$, P_0 is a positive definite matrix, so $det(Y)$ will always be a finite value. At time t^*, when $det(X(t^*))$ goes to zero, $det(P(t^*))$ will go to infinity, which means the Kalman Gain "K" will go to infinity. So we have to find the time t^* and applying a new method to prevent the Kalman Gain from going to infinity. In equation (77), $\Lambda_1 - \Lambda_2$ is the expression of the determinant of $X(t)$. At time t^*, the determinant of $X(t^*)$ equals 0, which is

$$det(X(t^*)) = \Lambda_1(t^*) - \Lambda_2(t^*) = 0 \tag{86}$$

$$0 < \frac{\sqrt{2qr}(q\pi_1 + r\pi_2)}{q^2r^2 - \pi_1\pi_2} < 2 \tag{87}$$

In equation (87), π_1 and π_2 are the initial values of P_{11} and P_{22}. q and r are standard deviations of the process noise and the measurement noise. When the values of these parameters satisfy this relationship, the time t^* is reached. In equation (85), when the determinant of P at t^* goes to infinity means that there will be at least one of the eigenvalues will be infinity. For the covariance matrix P, the two eigenvalues are λ_1 and λ_2.

If λ_2 goes to infinity.

$$det(P(t^*)) \to \infty \tag{88}$$

$$p(x) = \frac{1}{2\pi\sqrt{det(P)}} e^{\left(-\frac{1}{2}\begin{bmatrix}\tilde{x}_1 & \tilde{x}_2\end{bmatrix} P^{-1} \begin{bmatrix}\tilde{x}_1 \\ \tilde{x}_2\end{bmatrix}\right)} \tag{89}$$

Equation (89) shows the probability density $p(x)$ of $\tilde{x}_1$ *and* $\tilde{x}_2$ with zero mean. The lateral image of it is shown in Figure 13.

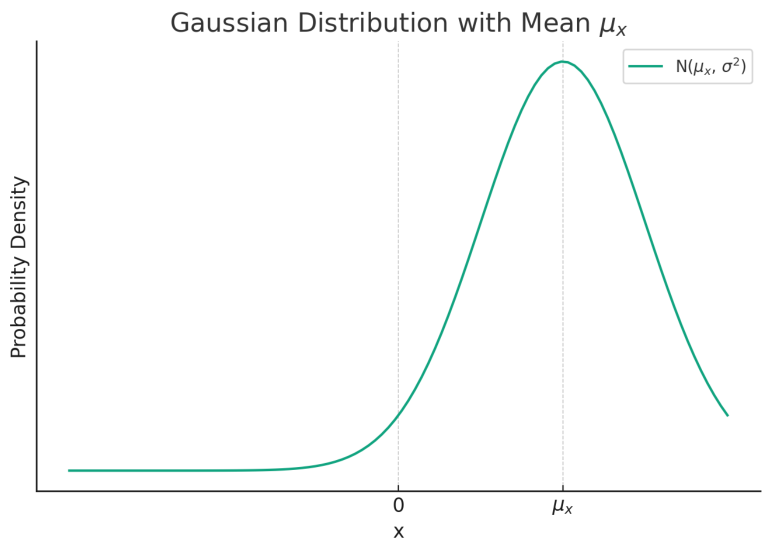

Fig. 13. $p(x)$

P is expressed in equation (90), λ_1 and λ_2 are eigenvalues of P, v_1 and v_2 are eigenvectors of P.

$$P = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T,\ v_1 v_1^T = 1,\ v_1 v_2^T = 0,\ v_2 v_2^T = 1 \tag{90}$$

If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, P^{-1} will be

$$P^{-1} = \frac{1}{\lambda_1} v_1 v_1^T + \frac{1}{\lambda_2} v_2 v_2^T \tag{91}$$

If and only if λ_1 or λ_2 goes to infinity, now suppose $\lambda_2 \to \infty$. The probability density function is shown in equation(93).

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \tilde{\xi} \tag{92}$$

$$p(\xi) = \frac{1}{2\pi\sqrt{\lambda_1}} e^{-\frac{(\tilde{\xi}^T v_1)^2}{2\lambda_1}} \frac{1}{\sqrt{\lambda_2}} e^{-\frac{(\tilde{\xi}^T v_2)^2}{2\lambda_2}} \tag{93}$$

In equation (93), when $\lambda_2 \to \infty$, the part containing λ_2 will be a Dirac delta function $\delta(v_2^T \tilde{x})$ function. The probability of it is 1.

So

$$p(\xi) = \frac{1}{2\pi\sqrt{\lambda_1}} e^{-\frac{(\tilde{\xi}^T v_1)^2}{2\lambda_1}} \cdot 1 \tag{94}$$

The equation (94) means that the probability of $v_1^T \tilde{x}$ is $N(0, \lambda_1)$ and with probability one that $v_2^T \tilde{x}$ equals zero.

The probability distribution of the ξ 2-vector can be plotted in the ξ-plane. It is Gaussian centered at the mean value in the ξ-plane. When λ_2 tends to zero, then the bell shaped Gaussian part becomes a line passing through the mean value point. The probability is zero for any point not on the line, Therefore only a linear combination of the estimates $\hat{x}_1$ and $\hat{x}_2$ is known with probability one. This is the cause of the finite escape of the Kalman gain, as will be explained in the following development.

6 Example 3: Linear Oscillator

6.1 Case A: Damped Harmonic Oscillator

The steady state of $P(t)$ is positive definite and constant, at the values obtained in page 213 of Grewal & Andrews [3]. Call this value P_∞. Call $P(0) = P_0$. If $P_\infty - P_0$ is positive definite, then P_0 is "smaller" than any $P(t)$ such that $P(t)$ will always decay to P_∞ from some "larger" initial condition P_0. But if $P_\infty - P_0$ is not positive definite, then $P(t)$ will "increase" from P_0 to P_∞. Therefore the error variance of the state vector can not decrease from P_0 with each successive measurement instant. The state variables are not known with certainty even with an infinite amount of data, but the variance decreases if P_0 is smaller than P_∞.

6.2 Case B: Undamped Oscillator Steady State

From page 214 of [3]: For the undamped oscillator, the damping coefficient $\tau \to \infty$. Then page213 [3] shows:

$$F = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \ H = \begin{pmatrix} 1 & 0 \end{pmatrix}, \ Q = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \tag{95}$$

The three scalar equations at the bottom of page 213 in [3].

$$0 = -P_{11}^2 + 2RP_{12} \tag{96}$$

$$0 = -RP_{11} + RP_{22} - P_{11}P_{12} \tag{97}$$

$$0 = -P_{12}^2 - 2R\omega^2 P_{12} + Rq \tag{98}$$

Solving these three equations can be done algebraically.

$$0 = P_{12}^2 + 2R\omega^2 P_{12} - Rq \text{ ,} is\ quadratic\ in\ P_{12}\ only \tag{99}$$

$$Then\ P_{12} = -R\omega^2 \pm \sqrt{(R\omega^2)^2 + Rq4} = -R\omega^2(1 \pm \sqrt{1 + \frac{4q}{R\omega^4}}) \tag{100}$$

$$P_{11}^2 = 2RP_{12} \Longrightarrow P_{11} = \pm\sqrt{2RP_{12}},\ P_{11} = \pm j\sqrt{2R(P_{12})}\ if\ P_{12} < 0 \tag{101}$$

$$P_{22} = P_{11} + \frac{P_{12}}{R} P_{11} \tag{102}$$

For $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix}$ to be real and positive definite, $P_{12} > 0$ and is real. This assumes a real value of $\lim_{t\to\infty} P(t)$ exists and is constant. The damping on ω, q, R, and initial conditions P_0 show that this is not always true. The Heisenberg uncertainty principle applies as in the double integrator. Then, the equations to find t^* become even more complicated. Further study of the undamped oscillator is beyond the scope of this work. Only the damped oscillator can be discussed here.

7 Unscented Kalman Filter(UKF)

The Taylor Expansion has been used to convert the a non-linear system to a linear system in the Extended Kalman Filter(EKF), but using the Taylor Expansion will cause non-linear errors. And the Jacobin matrix is not easy to implement, which increases the computational complexity of the algorithm. Compared with the Extended Kalman Filter, the Unscented Kalman Filter use unscented transform to solve the non-linear transfer problem [5].

In this example, we choose the linear oscillator as our dynamic system.

System Model:

$$x(k+1) = \Phi(\Delta)x(k) + \Phi(\Delta)\begin{bmatrix} 0 \\ 1 \end{bmatrix}[v(k+1) - v(k)]\Delta \tag{103}$$

In equation (103), the Δ is the time interval.

Measurement Model:

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + w(k) \tag{104}$$

Noise Model:

$$\begin{cases} v(k) & \sim N(0, q^2),\ q = 1 \\ w(k) & \sim N(0, r^2),\ r = 0.2 \end{cases} \tag{105}$$

In noise model, v and w are process noise and measurement noise, both of v and w conform to a Gaussian distribution.

$$\Phi(\Delta) = \begin{bmatrix} \cos\omega\Delta & \sin\omega\Delta \\ -\sin\omega\Delta & \cos\omega\Delta \end{bmatrix}, \ \omega = 2\pi f, \ f = 50Hz \tag{106}$$

Initial Conditions:

$$\begin{cases} x(0) & = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \hat{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ y(0) & = 3 \\ P(0) & = I_2 \end{cases} \tag{107}$$

Equation (107) defines initial values of x, $\hat{x}$, y and P. I_2 is a 2×2 identity matrix. In Figure 14, the process to implement the UKF is shown in Figure 14. The explanation of it is shown below .

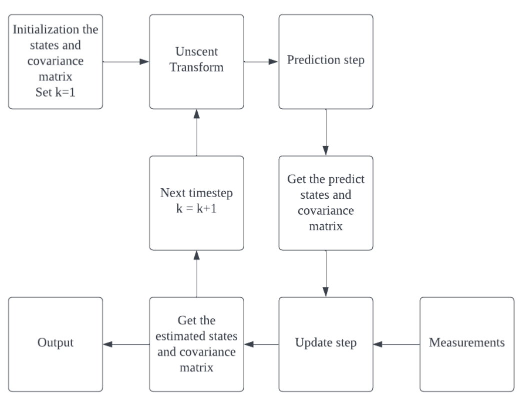

Fig. 14. Block diagram

Unscented transform for symmetric distribution sampling, λ is not eigenvalue, it has been redefined in [5]:

$$X^{(0)} = \hat{x},\ i = 0 \tag{108}$$

$$X^{(i)} = \hat{x} + (\sqrt{(n+\lambda)P_{X_i}})_i, i = 1,2,...n \tag{109}$$

$$X^{(i)} = \hat{x} - (\sqrt{(n+\lambda)P_{X_i}})_i, i = n+1, n+2,...2n \tag{110}$$

The first step of the symmetric distribution sampling is calculating the simga points. In equation (108), $X^{(0)}$ is the center point. In equation (109) and equation (110), the $\sqrt{(n+\lambda)P_{X_i}}$ represents the i-th column obtained from the squared of $(n+\lambda)P$ [5]. n is the dimension of the system. According to the model of the Linear Oscillator, n is 2. So there are five sigma points. λ is an adjustable parameter used to control the degree of dispersion of the point set. The sigma points are shown below:

$$X^{(0)} = \hat{x} \tag{111}$$

$$X^{(i)} = \hat{x} + (\sqrt{(2+\lambda)P_{X_i}})_i, i = 1,2 \tag{112}$$

$$X^{(i)} = \hat{x} - (\sqrt{(2+\lambda)P_{X_i}})_i, i = 3,4 \tag{113}$$

The next step is obtaining the weight of every sigma point. The method to obtain the weight is shown below [5].

$$\begin{cases} \omega_m^{(0)} & = \frac{\lambda}{n+\lambda} \\ \omega_m^{(i)} & = \frac{1}{2(n+\lambda)}, i = 1,2,...2n \end{cases} \begin{cases} \omega_c^{(0)} & = \frac{\lambda}{n+\lambda} + (1-\alpha^2+\beta) \\ \omega_c^{(i)} & = \frac{1}{2(n+\lambda)}, i = 1,2,...2n \end{cases} \tag{114}$$

with λ set at 1, the weights for the sigma points are:

$$\begin{cases} \omega_m^{(0)} & = \frac{1}{3} \\ \omega_m^{(i)} & = \frac{1}{6}, i = 1, 2, ...4 \end{cases} \begin{cases} \omega_c^{(0)} & = \frac{1}{3} + (1 - \alpha^2 + \beta) \\ \omega_c^{(i)} & = \frac{1}{6}, i = 1, 2, ...4 \end{cases} \tag{115}$$

In Polar Coordinates:

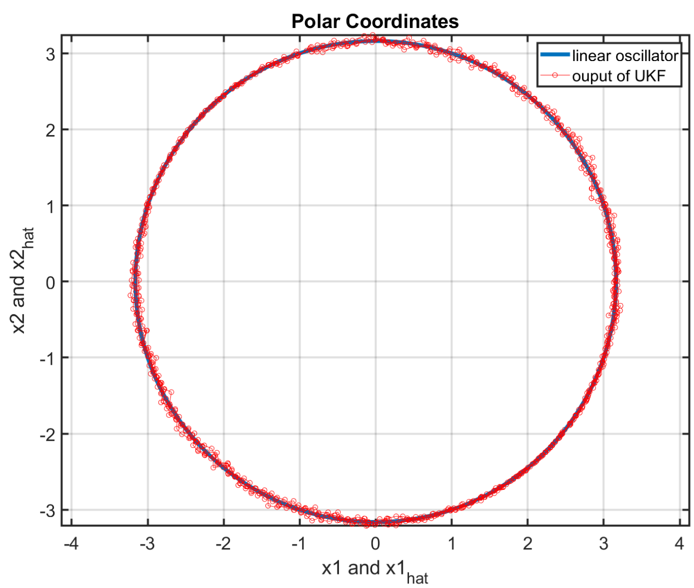

Fig. 15. Unscented Kalman Filter output vs linear oscillator output

In Figure 15, the blue line in the graph represents the theoretical trajectory of a linear oscillator, while the red circles indicate the results obtained from state estimation using the Unscented Kalman Filter (UKF). If the red circle points closely follow the blue line, it demonstrates that the Unscented Kalman Filter is performing well in estimating the system states. The distribution of these points also illustrates the measurement or estimation errors during the filtering process. Overall, the graph displays the effectiveness of the Unscented Kalman Filter in tracking the states of a linear oscillatory system.

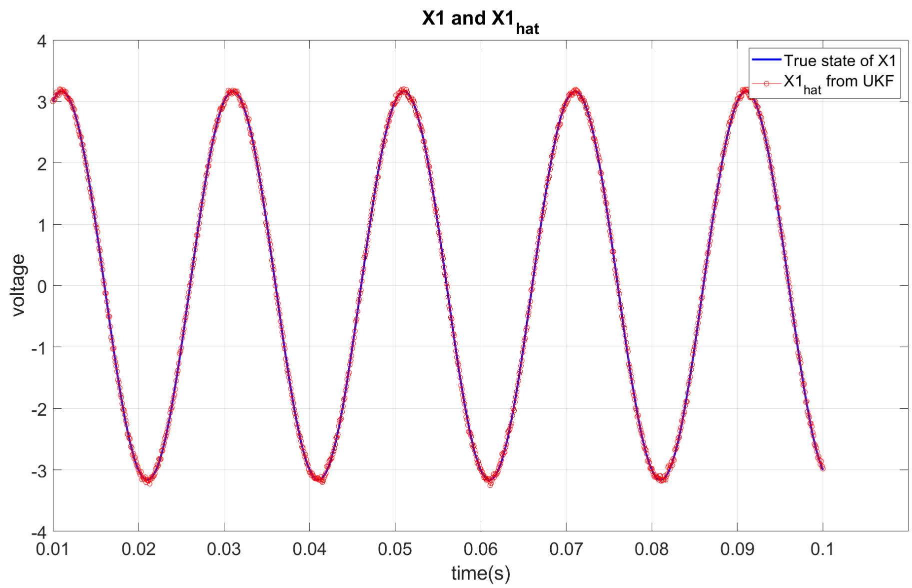

Fig. 16. UKF output voltage vs linear oscillator voltage

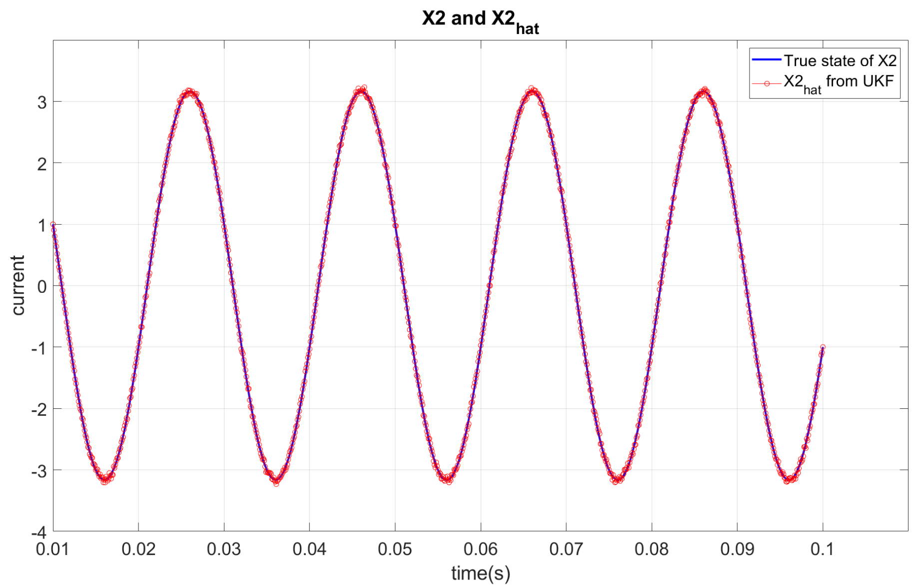

Fig. 17. UKF output current vs linear oscillator current

Figure 17 shared illustrates the true state of x_1 as a function of time, overlaid with the estimated state $\hat{X}_1$ derived from the Unscented Kalman Filter (UKF). The x-axis shows the time in seconds, and the y-axis represents the voltage. The true state is indicated by the solid blue line, and the Unscented Kalman Filter estimate is represented by red circles along the trajectory. The graph shows that the Unscented Kalman Filter estimates closely match the true state throughout the time period displayed, with the red circles frequently overlapping the blue line, indicating a high degree of accuracy in the estimation process by the Unscented Kalman Filter. The Figure 4 shows the relationship between x_2 and $\hat{x}_2$.

8 Conclusions

A cartoon in the New Yorker magazine by B. Sailer [6] showed an executive facing his employee, who says "better than the answer, I have a tremendous amount of data". If "the answer" is truth with probability one, the employee is correct only under certain conditions. These conditions are that the data is infinite, unbiased, unimodal, and regular, and the system is linear, and that the initial conditions and the inputs are known with certainty. The traditional proportional, integral, derivative (PID) feedback control design must incorporate a single integrator into the model to accommodate changes in a constant input. If this is linear, the residue of the integration term accounts for the variance in the steady state error. Here is given a method to determine this variance for a single integrator in general. Also, the double integrator and linear harmonic oscillator are investigated. These show that the state variance matrix can become singular in finite time(finite escape of the Kalman gain) under certain conditions. In this case, the feedback gain becomes infinite. Then, the Kalman gain must be reduced at that finite time. At this escape time, any application in which the Kalman filter is used must be modified appropriately.

The robustness of the Kalman filter applied to the linear oscillator is also a problem. The unscented Kalman filter is shown to mitigate this problem.

The solution to the error is the Riccati matrix differential equation, denoted $P(t)$ which sometimes becomes singular even for positive definite initial values. In Example 2, the double integrator, at least one of the eigenvector of $P(t)$, say $\lambda_2(t)$, can become zero at time t^* that is finite. Here, the conditions are found on the parameters and initial conditions for which $\lambda_2(t^*) = 0$.

This phenomenon must exist for higher-order dynamic systems corresponding to poles on the imaginary axis. Further research is needed to investigate this, because analytic solutions using the Hamiltonian are no longer possible for dynamic systems of higher order. It is shown that when both measurement and process noise are present, wherein the error variance will never be zero, and the estimate will always be uncertain. This is a generalization of the Heisenberg uncertainty principle. Finite escape of the time-varying Kalman gain can occur in any noisy dynamic system,. Even with an infinite amount of data, it would not be possible to attain certain knowledge of the true values of all the state variables of a noisy dynamic system.

Finite escape of the Kalman gain occurs when the $X(t)$ and $Y(t)$ $n \times n$ matrices in $P(t)X(t) = Y(t)$ no longer permits $P(t)$ to be positive definite. This happens when $det(X(t)) = 0$ or $Y(t)$ is no longer finite. Appropriate piratical solutions must then be found.

Also, the Kalman Filter is not robust with respect to linear system parameter variations. For a linear oscillator, a possible solution has been shown to be Unscented Kalman filtering, rather than the usual Kalman filtering. Because when the frequency of the linear oscillator is changing, the usual Kalman filtering can go unstable.

9 Appendices

Listing 1. Code of the Plot of P(t) normalized by $cosh(\omega t)$

```
w = q/r;
syms t;
P_t = (q*r*tanh(w*t)+pi0)/(1+(pi0/(q*r))*tanh(w*t));
P_t_val = subs(P_t, t);
```

Listing 2. Code of the Kalman-Bucy Filter for single integrator

```
t_interval = 0.001;
tspan = 0:t_interval:0.2;

for i = 1:length(tspan)-1
% Kalman-Bucy Filter Update
dx_hat = P(i) * (y(i) - x_hat(i)) / r^2;
[~, dx_hat] = ode45(@(t, x_hat) dx_hat,
            [tspan(i) tspan(i+1)], x_hat(i));
x_hat(i+1) = dx_hat(end);

dP = q^2 - P(i)^2 / r^2;
[~, dP] = ode45(@(t, P) dP,
            [tspan(i) tspan(i+1)], P(i));
P(i+1) = dP(end);
```

```
    end
```

Listing 3. Code of the Kalman-Bucy Filter for double integrator

```
for i = 1:length(tspan)-1
    K1 = P11(i)/ r^2;
    K2 = P12(i)/ r^2;

    dX = @(t, X) [X(2) + K1 * (y(i) - X(1));   % dx1/dt
                  K2 * (y(i) - X(1))];          % dx2/dt
    [~, X_temp] = ode45(dX, [tspan(i) tspan(i+1)],
                        [x1_hat(i); x2_hat(i)]);

    dP = @(t, P) [2 * P(2) - P(1)^2 / r^2;  % dP11/dt
                  P(4) - P(1) * P(2) / r^2; % dP12/dt
                  P(4) - P(1) * P(2) / r^2; % dP21/dt
                  q^2 - P(2)^2 / r^2];      % dP22/dt

    P0 = [P11(i); P12(i); P21(i); P22(i)];
    [~, P_temp] = ode45(dP, [tspan(i) tspan(i+1)], P0);
end
```

10 Bibliography

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