Title

Nonadiabatic Berry Phase for a Quantum System with a Dynamical Semisimple Lie Group

Journal

Physical Review A, 42(9)

Author

Wang, S.-J.

Publication Date

1990-07-01

LBL-29331
Preprint

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Submitted to Physical Review A

Non-Adiabatic Berry's Phase for a Quantum System with a Dynamical Semisimple Lie Group

S.-J. Wang

July 1990

NUCLEAR SCIENCE DIVISION

Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098.

Non-adiabatic Berry's Phase For A Quantum System With A Dynamical Semi-simple Lie Group

Shun-Jin Wang

Nuclear Science Division, Lawrence Berkeley Laboratory

University of California, Berkeley, CA 94720, USA

And

Center of Theoretical Physics, CCAST (World Lab.) Beijing and

Department of Modern Physics, Lanzhou University, Lanzhou 730001, PR China

Abstract: Non-adiabatic Berry's phase is investigated for a quantum system with a dynamical semisimple Lie group within the framework of the generalized cranking approach. An expression for non-adiabatic Berry's phase is given, which shows that non-adiabatic Berry's phase is related to the expectation value of Cartan operators along the cranking direction in group space, and that it depends on i) the geometry of the group space, ii) the time evolution ray generated by the Hamiltonian(i.e., by the dynamics) in some irreducible representation Hilbert space and iii) the cranking rate. The expression also provides a simple algorithm for calculating the non-adiabatic Berry's phase. The general formalism is illustrated by examples of SU(2) dynamic group.

This work was supported in part by the Director, Office of Energy Research, Division of Nuclear Physics of the Office of High Energy and Nuclear Physics of the U.S.Department of Energy under Contract No.DE-AC03-76SF00098 and by the Natural Science Foundation of China.

Adiabatic Berry's phase [1] has been exploited extensively in a large number of theoretical and experimental articles[2], and much knowledge and deep insight have been obtained in this respect. However,although non-adiabatic Berry's phase has been addressed by Berry[3] and several other authors [4], a comparable deep insight is still lacking. Since non-adiabatic Berry's phase is related to dynamical effects, its study depends on specific dynamics, i.e., on the structure of the Hamiltonian. Thus the investigation of non-adiabatic Berry's phase is more difficult. From our previous studies , we found that the description of non-adiabaticity may become easier if a quantum system possesses a dynamical group. In our previous papers, three types of systems were investigated: a photon propagating in an optical helix[5], a spin particle in a rotating magnetic field [6], and a rotating deformed nucleus[7]. For all the three systems, the relevant dynamical group is the SU(2) group, and the problems are solved by the cranking method developed in nuclear physics [8]. Berry's phase is obtained analytically if the Hamiltonian is a linear function of SU(2) generators, and can be calculated straightforwardly even though the Hamiltonian is non-linear in the generators.It is found that for SU(2) dynamical group, Berry's phase is related to the expectation value of the spin and non-adiabatic effect on Berry's phase manifests itself as spin alignment. In this note, we generalize the above studies to a quantum system which possesses a dynamical semi-simple Lie group and exploit physical-geometrical aspects of non-adiabatic Berry's phase.

Consider a quantum system whose Hamiltonian is a function of generators of a semi-simple Lie group G,

$$\mathcal{H}_0 = \mathcal{H}_0(X_\mu) = \mathcal{H}_0(H_i, E_\alpha), \qquad (1)$$

where the generators $\{ X_\mu \}$ or $\{ H_i , E_\alpha \}$ in Cartan form obey standard commutator relations [9],

$$[H_i , H_j] = 0 , \quad i,j = 1,2,\ldots l, \qquad (2a)$$

$$[H_i , E_\alpha] = \alpha_i E_\alpha, \quad \alpha = 1,2,\ldots (n-1), \qquad (2b)$$

$$[\, E_\alpha \,,\, E_{-\alpha} \,] = \alpha^i H_i , \qquad (2c)$$

$$[\, E_\alpha \,,\, E_\beta \,] = N_{\alpha\beta}\, E_{\alpha+\beta} \,, \quad \text{if} \quad \alpha+\beta \neq 0 \,. \qquad (2d)$$

Where $\{ H_i \}$ is Cartan sub-algebra, $\{ E_\alpha , E_{-\alpha} \}$ are raising and lowering operators, l the rank of the group, n the order of the group.

First consider the simplest case where the Hamiltonian is a linear function of the generators. Generally,

$$\mathcal{H}_0 = \epsilon\, \vec{\beta}\cdot\vec{X} = \epsilon \sum \beta_\mu X_\mu = \epsilon \{ \sum \beta_\alpha E_\alpha + \sum \beta_i H_i \}, \qquad (3a)$$

where $\vec{\beta}$ is a vector in group parameter space,

$$\vec{\beta} = \{ \beta_\mu \} = \{ \beta_\alpha \,,\, \beta_i \} \,. \qquad (3b)$$

$\mathcal{H}_0$ can be rewritten as

$$\mathcal{H}_0 = \epsilon \exp\{-\sum (z_\alpha E_\alpha - z^*_\alpha E_{-\alpha})\}\, \vec{a}\cdot\vec{H} \exp\{+ \sum (z_\alpha E_\alpha - z^*_\alpha E_{-\alpha})\} \,, \qquad (4)$$

where

$$\vec{a}\cdot\vec{H} = \sum a_i H_i \,, \qquad \sum a_i^2 = 1, \quad \sum |\beta_\alpha|^2 + \sum \beta_i^2 = 1 \,, \qquad (5)$$

a_i and z_α(z^*_α) are functions of β_μ. Suppose $|m\rangle$ are eigenvectors of $\vec{H}$,

$$\vec{H}\, |m\rangle = \vec{m}\, |m\rangle \,, \qquad \vec{m} = \{ m_i \mid i=1..l \} \,. \qquad (6)$$

Now crank the system through a periodic time-dependent unitary transformation. The Hamiltonian of the system then becomes time-dependent,

$$\mathcal{H}(t) = \exp\{-i\vec{n}\cdot\vec{H}\omega t \}\, \mathcal{H}_0 \exp\{i\vec{n}\cdot\vec{H}\omega t \}$$

$$= \epsilon\, [\, \vec{\beta}_I(t)\cdot\vec{E} + \vec{\beta}_{II}\cdot\vec{H} \,], \qquad (7)$$

where

$$\vec{\beta}_I(t) = \{ \beta_\alpha \exp\{-i\vec{n}\cdot\vec{\alpha}\,\omega t \} \}, \qquad \vec{\beta}_{II} = \{ \beta_i \}, \qquad (8)$$

with

$$\vec{n}\cdot\vec{\alpha} = \sum n_i \alpha_i = \pm \text{integer} \,. \qquad (9)$$

Since $\vec{b}\cdot\vec{X}$ can be considered to be a Cantan operator or a combination of Cartan operators, $\exp\{-i\vec{n}\cdot\vec{H}\,\omega t\}$ is a general periodic time-dependent transformation in the group space. Thus the cranked Hamiltonian (7) is a general form.

The equation of motion for the cranked system is

$$i\frac{\partial\psi(t)}{\partial t} = \mathcal{H}(t)\,\psi(t) \quad . \tag{10}$$

Turn to the intrinsic frame through a unitary transformation,

$$\psi(t) = \exp\{-i\vec{n}\cdot\vec{H}\,\omega t\}\,\eta(t) \quad . \tag{11}$$

The equation of motion for $\eta(t)$ is

$$i\frac{\partial\eta(t)}{\partial t} = \mathcal{H}(\omega)\,\eta(t) \quad , \tag{12}$$

where the Routhian operator $\mathcal{H}(\omega)$, is defined as

$$\mathcal{H}(\omega) = \mathcal{H}_0 - \omega\,\vec{n}\cdot\vec{H} = \epsilon\,[\,\sum\beta_\alpha E_\alpha + \sum(\beta_i - \frac{\omega}{\epsilon}n_i)\,H_i\,]$$

$$= \bar{\epsilon}\,[\,\vec{\bar{\beta}}_{I}\cdot\vec{E} + \vec{\bar{\beta}}_{II}\cdot\vec{H}\,] \quad , \tag{13a}$$

which can be rewritten as

$$\mathcal{H}(\omega) = \bar{\epsilon}\exp\{-\sum(\bar{z}_\alpha E_\alpha - \bar{z}^*_\alpha E_{-\alpha})\}\;\vec{a}\cdot\vec{H}\,\exp\{\sum(\bar{z}_\alpha E_\alpha - \bar{z}^*_\alpha E_{-\alpha})\} . \tag{13b}$$

where a_i , $\bar{z}_\alpha$ and $\bar{z}^*_\alpha$ are functions of $\bar{\beta}_\mu$, and the renormalized parameters are

$$\bar{\epsilon} = \epsilon\gamma \tag{14}$$

$$\bar{\beta}_\alpha = \beta_\alpha/\gamma \quad , \qquad \bar{\beta}_i = (\beta_i - \frac{\omega}{\epsilon}n_i)/\gamma \quad , \tag{15a}$$

$$\gamma = [\,1 - 2\frac{\omega}{\epsilon}\sum\beta_i n_i + (\frac{\omega}{\epsilon})^2\sum n_i^2\,]^{1/2} \; . \tag{15b}$$

Solutions of eqs.(10) and (12) are

$$\eta(t) = \exp\{-i\mathcal{H}(\omega)t\}\,\eta(0) \quad , \tag{16}$$

$$\psi(t) = U(t)\,\psi(0) \quad . \tag{17}$$

Where the evolution operator is

$$U(t) = \exp\{-i\vec{n}\cdot\vec{H}\,\omega t\}\exp\{-i\mathcal{H}(\omega)t\} \quad . \tag{18}$$

Let us consider the eigen equations of $\mathcal{H}_0$ and $\mathcal{H}(\omega)$,

$$\mathcal{H}_0 \varphi_m = \epsilon_m \varphi_m , \tag{19}$$

$$\mathcal{H}(\omega) \eta_m = E_m \eta_m , \tag{20}$$

with the solutions

$$\epsilon_m = \epsilon \vec{a} \cdot \vec{m} = \epsilon \sum a_i m_i , \tag{21a}$$

$$\varphi_m = \exp\{-\sum (z_\alpha E_\alpha - z_\alpha^* E_{-\alpha}) \} |m\rangle ; \tag{21b}$$

and

$$E_m = \bar{\epsilon} \vec{a} \cdot \vec{m} = \bar{\epsilon} \sum a_i m_i , \tag{22a}$$

$$= \exp\{-\sum (\bar{z}_\alpha E_\alpha - \bar{z}_\alpha^* E_{-\alpha}) \} |m\rangle . \tag{22b}$$

Consider solutions after one period T ($T = 2\pi/\omega$). The evolution operator in one period is

$$U(T) = \exp\{-i\vec{n}\cdot\vec{H}\, 2\pi\} \exp\{-i\mathcal{H}(\omega)T\} . \tag{23}$$

Since

$$\exp\{-i\vec{n}\cdot\vec{H}\, 2\pi\} \mathcal{H}(\omega) \exp\{i\vec{n}\cdot\vec{H}\, 2\pi\} = \mathcal{H}(\omega) , \tag{24}$$

U(T) and $\mathcal{H}(\omega)$ commute and have common eigen vectors, i.e.,

$$U(T)\eta_m = \exp\{-i\Phi_m\} \eta_m , \tag{25}$$

where the total phase Φ_m will be given later.

Consider cyclic or recurrent solutions whose initial states are eigen-states of $\mathcal{H}(\omega)$,

$$\psi_m(0) = \eta_m . \tag{26}$$

After one period,

$$\psi_m(T) = \exp\{-i\vec{n}\cdot\vec{H}\, 2\pi\} \exp\{-i\mathcal{H}(\omega)T\} \eta_m$$

$$= \exp\{-iE_m T - i2\pi\, \vec{n}\cdot\vec{m}\} \psi_m(0) . \tag{27}$$

The total phase is

$$\Phi_m = E_m T + 2\pi \vec{n}\cdot\vec{m} \quad . \qquad (28)$$

The expectation value of $\mathcal{H}(t)$ is

$$\mathcal{E}_m(t) = \langle \Psi_m(t) | \mathcal{H}(t) | \Psi_m(t) = \langle \eta_m | \mathcal{H}_o | \eta_m \rangle$$

$$= E_m(\omega) + \omega \langle \eta_m | \vec{n}\cdot\vec{H} | \eta_m \rangle = E_m(\omega) + \omega\, \vec{n}\cdot\langle \vec{m} \rangle \quad , \qquad (29)$$

where

$$\langle \vec{m} \rangle = \langle \eta_m | \vec{H} | \eta_m \rangle \, . \qquad (30)$$

From eq.(29) we obtain the dynamical phase

$$\Phi_m^d = \int_0^T \mathcal{E}_m(t)\, dt = E_m(\omega) T + 2\pi\, \vec{n}\cdot\langle \vec{m} \rangle \; , \qquad (31)$$

and Berry's phase

$$\Phi_m^b = -\left(\Phi_m - \Phi_m^d \right) = -2\pi\, \vec{n}\cdot\vec{m} \left(1 - \vec{n}\cdot\langle \vec{m} \rangle / \vec{n}\cdot\vec{m} \right)$$

$$= -2\pi\, \vec{n}\cdot\vec{m} \left(1 - \langle \eta_m | \vec{n}\cdot\vec{H} | \eta_m \rangle / \vec{n}\cdot\vec{m} \right) \, . \qquad (32)$$

Eq.(32) indicates that Berry's phase is related to the expectation value of Cartan operators along the cranking $\vec{n}$ -direction and depends on i) the geometry of the group space where the vectors $\vec{n}$ and $\vec{m}$ reside, ii) the ray or η_m generated by the Hamiltonian (dynamics), and iii) the cranking rate ω . The expression (32) also provides an algorithm for calculating non-adiabatic Berry's phase, since, given an irreducible representation of the dynamical group, the calculation of eigen-vectors η_m and expectation value $\langle \eta_m | \vec{n}\cdot\vec{H} | \eta_m \rangle$ is straightforward.

Now consider the general cases where the Hamiltonian is a non-linear function of the group generators,

$$\mathcal{H}_o = \mathcal{H}_o(\beta_\alpha E_\alpha \; , \; \beta_i H_i) \; . \qquad (33)$$

After cranking, the Hamiltonian becomes

$$\mathcal{H}(t) = \mathcal{H}_o(\beta_\alpha(t) E_\alpha \; , \; \beta_i H_i) \; . \qquad (34)$$

The expression (32) of Berry's phase is also applicable for the non-linear case. However, the eigen-solutions of $\mathcal{H}(\omega)$ have to be obtained by a straightforward numerical calculation.

In what follows we give examples to illustrate the above general formalism. Consider the SU(2) dynamical group which, as we mentioned before, is of practtical and theoretical interest. For the linear case, the Hamiltonian is assumed to be[6]

$$\mathcal{H}_0 = \vec{\Omega}\cdot\vec{J} = \Omega \exp\{-\theta(J_+ - J_-)\}\, J_z \exp\{\theta(J_+ - J_-)\}\ , \qquad (35)$$

which describes a spin particle in a magnetic field. In the above

$$\vec{a}\cdot\vec{H} = J_z\ , \qquad (36a)$$

$$\vec{\Omega} = \Omega\,(\sin\theta\ ,\ 0\ ,\ \cos\theta\ )\ . \qquad (36b)$$

The cranked Hamiltonian is

$$\mathcal{H}(t) = \exp\{-iJ_z\omega t\}\,\mathcal{H}_0 \exp\{iJ_z\omega t\} = \vec{\Omega}(t)\cdot\vec{J}\ , \qquad (37a)$$

$$\vec{\Omega}(t) = \Omega\,(\sin\theta\ \cos\omega t,\ \sin\theta\ \sin\omega t,\ \cos\theta\ ), \qquad (37b)$$

which indicates that the magnetic field precesses about the z-axis with frequency ω . The Routhian operator and its eigen-solutions are

$$\mathcal{H}(\omega) = \mathcal{H}_0 - \omega J_z = \vec{\bar{\Omega}}\cdot\vec{J}$$

$$= \bar{\Omega} \exp\{-\bar{\theta}(J_+ - J_-)\}\, J_z \exp\{+\bar{\theta}(J_+ - J_-)\}\ , \qquad (38a)$$

$$\vec{\bar{\Omega}} = \bar{\Omega}\,(\sin\bar{\theta}\ ,\ 0\ ,\ \cos\bar{\theta}\ )\ , \qquad (38b)$$

$$\eta_m = \exp\{-i\,\bar{\theta}\,J_y\}\,|m\rangle\ , \qquad (38c)$$

$$E_m = m\,\bar{\Omega}\ , \qquad (38d)$$

$$\bar{\Omega} = \Omega\gamma\ ,\quad \gamma = [\,1 - 2\frac{\omega}{\Omega}\cos\theta + (\frac{\omega}{\Omega})^2\,]^{1/2}\ . \qquad (38e)$$

The Berry's phase is

$$\Phi_m^b = -\,2m\pi\,(\,1 - \langle\eta_m|J_z|\eta_m\rangle/m\,) \qquad (39a)$$

$$= -\,2m\pi\,(\,1 - \cos\bar{\theta}\ )\ , \qquad (39b)$$

where

$$\cos\bar{\theta} = (\cos\theta - \omega/\Omega) / \gamma \quad . \tag{39c}$$

For the non-linear case, the Hamiltonian is assumed to be

$$\mathcal{H}_0 = (\vec{\Omega}\cdot\vec{J})^2 \tag{40a}$$

and

$$\mathcal{H}(t) = [\vec{\Omega}(t)\cdot\vec{J}]^2 . \tag{40b}$$

which are used to describe nuclear quadrupole resonances [10] . The Berry's phase takes the same form as eq.(39a). However the eigen-solutions of $\mathcal{H}(\omega)$ have to be calculated numerically.

In conclusion, we have generalied the investigation of non-adiabatic Berry's phase of a quantum system with SU(2) dynamic group to a quantum system with any dynamic semisimple Lie group within the framework of the cranking approach. The non-adiabatic Berry's phase is given in terms of the expectation value of Cartan operators, which provides a simple algorithm for calculating non-adiabatic Berry's phase and gives Berry's phase a physical-geometric interpretation, since the expectation value of Cartan operators in a quantum system has both physical -geometric meanings. The illustrations of the SU(2) examples indicate that the above formalism is useful.

The author thanks Professor W.J.Swiatecki for illuminating discussions. This work was supported in part by the Director, Office of Energy Research, Division of Nuclear Physics of the Office of High Energy and Nuclear Physics of the U.S. Department of Energy under Contract No.DE-AC03-76SF00098 and by the Natural Science Foundation of China.

References

1. M.V.Berry, Proc.Roy.Lond.A392(1984)45.

2. A.shapere and F.Wilczek, Geometric Phase in Physics, Advances Series in Mathematics Vol.5, World Scientific, Singapore, 1989.

3. M.V.Berry, Nature, Vol.326(1987)277.

4. Y.Aharonov and R.Anandan, Phys.Rev.Lett.58(1987)1593;
J.Samuel and R.Bhamdari, Phys.Rev.Lett.60(1988)2339;
A.Bulgac, Phys.Rev.A37(1988)4084.

5. S.J.Wang, LBL Preprint- , May, 1990; submitted to Phys.Rev.A.

6. S.J.Wang, LBL Preprint- , July, 1990; submitted to Phys.Rev.A.

7. S.J.Wang, et al., in preparation.

8. R.Bengtsson and J.D.Garrett, Lund Mph-84/18, 1984; to be published by World Scientific Publishing Co., Singapore.

9. B.G.Wybourne, Classical Groups For Physicists, (Wiley, New York), 1974.

10. R.Tycko, Phys.Rev.Lett.58(1987)2281;
D.Suter, et al., Phys.Rev.Lett.60(1988)1218.