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Linear Optimal Regulation to Zero Dynamics

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***Abstract*—A general problem encompassing output regulation and pattern generation can be formulated as the design of controllers to achieve convergence to a persistent trajectory within the zero dynamics on which an output vanishes. We develop an optimal control theory for such design by adding the requirement to minimize the H_2 norm of a closed-loop transfer function. Within the framework of eigenstructure assignment, the optimal control is proven identical to the standard H_2 control in form. However, the solution to the Riccati equation for the linear quadratic regulator is not stabilizing. Instead it partially stabilizes the closed-loop dynamics excluding the zero dynamics. The optimal control architecture is shown to have the feedback of the deviation from the subspace of the zero dynamics and the feedforward of the control input to remain in the subspace.**

I. Introduction

A fundamental problem in the systems and control field is the output regulation, where the objective is the design of a feedback controller that makes a selected output converge to zero in response to persistent command and/or disturbance inputs [1], [2]. The output regulation has been extensively studied in the literature for a general class of nonlinear systems driven by an exogenous system [3], [4]. The problem is essentially equivalent to achieving convergence to a trajectory defined by the zero dynamics associated with the output being regulated [3]. For linear systems, zero dynamics are characterized by the regulator equation [5], which provides a condition for solvability of the problem [1], [6].

Closely related to the output regulation is the eigenstructure assignment, where a controller is designed to assign a prescribed set of eigenvalues and eigenvectors to the closed-loop system [7]–[9]. The linear output regulation is a special case of the eigenstructure assignment, where the exosystem dynamics embedded in the generalized plant is preserved in the closed-loop eigenstructure [6]. In general, the eigenstructure assignment does not necessarily require the exosystem, and can also be used as a framework for *autonomous* pattern generation. For instance, coordinated motion pattern of multiagent systems can be achieved by embedding an eigenstructure in the network [10], rather than driving each agent by a network of exosystems that generate reference commands as in [11].

For linear systems, the best result on optimal output regulation to date appears to be the one in [12] where, in the spirit of classical linear quadratic regulator (LQR), the L_2 norm of an error is minimized in response to the initial state. In this formulation, the sensor noise is generated through the exosystem dynamics, rendering the optimal control problem singular. As a result, the controller construction is nontrivial and left to the singular optimal control theory. More recently, the problem of optimal pattern generation has been solved [13], where the controller is designed to make state variables converge to a prescribed pattern (oscillation, constant, or combination) while minimizing a transient cost function. This result assumes stabilizability of the generalized plant, hence exosystems (e.g. disturbance model) cannot be augmented as in the output regulation paradigm.

In this paper, we develop a theory that unifies the output regulation and autonomous pattern generation within the eigenstructure framework. In particular, we consider a plant with anti-stable zero dynamics, and solve an optimal control problem to achieve convergence to a persistent trajectory within the zero dynamics while minimizing the H_2 norm of a closed-loop transfer function. The general formulation captures optimal output regulation and pattern generation problems, with or without the exosystem. Unlike [13], we do not assume stabilizability of the generalized plant, allowing for augmentation of exosystems if desired. Unlike [12], we impose standard regularity assumptions on the generalized plant [14] so that a simple state space formula can be obtained for the optimal control. The formula is analogous to the standard theory [14], but is based on the *partially* stabilizing solution to the Riccati equation associated with the zero dynamics.

Notation: Bold face letters denote state space systems: $y = \mathbf{G}u$ means that the input u and output y are related by

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix},$$

where (A, B, C, D) are real constant matrices specific to $\mathbf{G}$, and x is the state vector. For a transfer function $\mathcal{H}(s)$, the H_2 norm is denoted by $\|\mathcal{H}\|_2$. For signals α and β, notation $\alpha \to \beta$ means $\|\alpha(t) - \beta(t)\|$ approaches zero as t goes to $+\infty$. The spectrum of matrix M is denoted by $\mathrm{eig}(M)$. We write $\mathrm{col}(M_1, M_2)$ and $\mathrm{row}(M_1, M_2)$ to mean the matrices obtained by stacking the arguments in a column and row, respectively. The set of complex numbers with negative real parts is denoted by $\mathbb{C}_-$. For a tall full column rank matrix X, we denote by $(X^-, X_\perp, X_\perp^-)$ a matrix triple such that

$$\begin{bmatrix} X_\perp \\ X^- \end{bmatrix} \begin{bmatrix} X_\perp^- & X \end{bmatrix} = I, \quad W := \begin{bmatrix} X_\perp^- & X \end{bmatrix}, \qquad (1)$$

where W is a square matrix.

We denote by $\mathbb{S}$ the set of matrices (A, B, C, D) such that (A, B) is stabilizable, $(\hat{C}, \hat{A})$ has no unobservable mode on the imaginary axis, and $D^\mathsf{T} D = I$, where $\hat{A} := A - BD^\mathsf{T} C$ and $\hat{C} := (I - DD^\mathsf{T})C$. We denote the "dual" of $\mathbb{S}$ by $\mathbb{S}^\mathsf{T}$, which is the set of matrices (A, B, C, D) such that $(A^\mathsf{T}, C^\mathsf{T}, B^\mathsf{T}, D^\mathsf{T}) \in \mathbb{S}$. Conditions $(A, B, C, D) \in \mathbb{S}$ and $(A, B, C, D) \in \mathbb{S}^\mathsf{T}$ are standard in optimal control theories [5], [14] to guarantee existence of the stabilizing solutions to the Riccati equations associated with the LQR and Kalman filter, respectively.

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II. Problem Statement

A. General Problem: Regulation to Zero Dynamics

Consider the generalized plant $\mathbf{G}$ described by

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_1 u, \\ y &= C_2 x + D_2 w, \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $y(t) \in \mathbb{R}^{n_y}$ is the sensor output available for feedback control, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous input possibly including reference commands, disturbances, and sensor noises, and $z(t) \in \mathbb{R}^{n_z}$ is the performance output we wish to keep small.

We will formulate an optimal control problem for (2) to achieve convergence to a trajectory within the zero dynamics. For state space system (A, B_2, C_1, D_1) described by (2) with $w = 0$, we define the zero dynamics as the collection of trajectories (x, u) such that $z = 0$. When D_1 has a full column rank, $z = 0$ implies

$$\dot{x} = (A - B_2 D_1^- C_1)x, \quad u = -D_1^- C_1 x.$$

Hence, the zero dynamics is contained in the set of trajectories (x, u) satisfying these equations. A subset is given by trajectories residing in an invariant subspace, i.e.,

$$x = Xe^{\Lambda t}\eta_o, \quad u = Ue^{\Lambda t}\eta_o, \tag{3}$$

with arbitrary vector η_o, where matrix triple (X, U, Λ) satisfies

$$X\Lambda = (A - B_2 D_1^- C_1)X, \quad U = -D_1^- C_1 X. \tag{4}$$

This subset is contained in the zero dynamics if and only if $C_1 X + D_1 U = 0$ holds, which enforces $z = 0$. It can readily be verified that (X, U, Λ) satisfies this condition and (4) if and only if the following regulator equations are satisfied:

$$AX + B_2 U = X\Lambda, \tag{5a}$$

$$C_1 X + D_1 U = 0. \tag{5b}$$

To make a formal problem statement, let us introduce:

Assumption 1: The regulator equations (5) are satisfied, and X is a tall matrix with a full column rank.

We fix such triple (X, U, Λ) and consider the control design to achieve convergence to (3) for some η_o. We then impose

Assumption 2: $\mathrm{eig}(\Lambda) \cap \mathbb{C}_- = \emptyset$

without loss of generality since any stable invariant subspace does not contribute to the steady state behavior and can be removed. The problem is the following:

Problem 1: Consider the generalized plant (2). Let matrices (X, U, Λ) be given and suppose Assumptions 1 and 2 are satisfied. Design a controller $u = \mathbf{K}y$ that makes the closed-loop state converge to the zero dynamics in an optimal manner, with the following specifications:

(s1) Consider the closed-loop system with $w = 0$. For each η_o, there exists an initial state such that

$$x(t) \to Xe^{\Lambda t}\eta_o, \quad u(t) \to Ue^{\Lambda t}\eta_o. \tag{6}$$

For each initial state, there exists η_o such that (6) holds.

(s2) The closed-loop transfer function from w to z, denoted by $\mathcal{H}(s)$, is stable and its H_2 norm is the minimum among those achieved by any controller satisfying (s1).

Unlike classical optimal control theories, our control design does not require internal stability of the closed-loop system. Instead, we require that the plant state x converge to a nonzero trajectory $Xe^{\Lambda t}\eta_o$ defined by the anti-stable zero dynamics characterized in (5). Yet, property (6) with (5b) implies that the output z converges to zero, and the convergence should be optimal as dictated by the minimum H_2 norm of the closed-loop transfer function $\mathcal{H}(s)$.

As is well known, the H_2 cost is equivalent to the L_2 norm of z in response to the impulse input $w(t) = w_o\delta(t)$, square-averaged over all directions of the vector w_o, where $\delta(t)$ is the Dirac delta function. In this context, setting the initial condition is a role of w. Another interpretation of the H_2 norm is the maximum peak value (L_∞ norm) of the output in response to the class of square-integrable (L_2) inputs. In this case, w is an L_2 signal and its role is to drive the state over time. In either case, convergence (6) is achieved and z decays to zero in the H_2 optimal manner.

Assumption 1 does not necessarily restrict the class of physical plants for which the theory developed in this paper is applicable. This is because the output z is chosen by the designer for the purpose of optimization, and the zero dynamics may not be inherent with the physical system. In the subsequent sections, we will provide specific contexts in which Problem 1 arises and justify its significance.

B. Output Regulation Problem

This section introduces the general output regulation problem and then shows that it is a special case of Problem 1. Consider the plant

$$\begin{aligned} \dot{x}_p &= A_o x_p + B_o u + E_o \eta + G_o w, \\ e &= C_e x_p + D_e u + F_e \eta, \\ y &= C_2 x + D_2 w, \quad x := \mathrm{col}(x_p, \eta), \end{aligned} \tag{7}$$

where x_p is the state, u is the control input, y is the measured output available for the controller, w is the exogenous input, e is the error output to be regulated, and η is the signal that can be modeled by an exogenous system

$$\dot{\eta} = \Lambda\eta + H_o w. \tag{8}$$

The general description of the plant allows for w to contain various sources of exogenous signals such as the disturbance, reference command, and sensor noise, each of which can be persistent or non-persistent, with or without the knowledge of their dynamics. For example,

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \quad \begin{bmatrix} G_o \\ H_o \\ D_2 \end{bmatrix} = \left[\begin{array}{ccc} * & 0 & 0 \\ \hline 0 & * & 0 \\ \hline 0 & 0 & * \end{array}\right] \tag{9}$$

would generate signal η with known dynamics Λ by w_2 through (8), and capture the disturbances and sensor noises with unknown dynamics by w_1 and w_3, respectively. Assumption 2 means that η is persistent, which can be assumed without loss of generality by absorbing any decaying components of Λ dynamics into A_o and making them driven by part of w_1. In this case, Λ would typically have eigenvalues on the imaginary axis to generate nonzero constant or sinusoidal η using impulse input w_2, effectively setting the initial state $\eta(0)$ through H_o.

The output regulation problem [1] is to design a feedback controller $u = \mathbf{K}y$ such that (a) the closed-loop system is internally stable except for the exosystem dynamics Λ, and (b) the error converges to zero, $e(t) \to 0$, for arbitrary initial state when $w = 0$. It is well known [1], [6] that, under mild assumptions on stabilizability and detectability, the output regulation problem is solvable if and only if there exist matrices (X_p, U) with the following property.

Assumption 3: The following regulator equations hold:

$$A_o X_p + B_o U + E_o = X_p \Lambda, \tag{10a}$$

$$C_e X_p + D_e U + F_e = 0. \tag{10b}$$

The equations (10) essentially mean that the solution of the form $x = X_p\eta$ and $u = U\eta$ exists for (7) when driven by $w = 0$ and η satisfying $\dot{\eta} = \Lambda\eta$, and the output e vanishes on the solution. In this case, every output regulator achieves

$$x_p \to X_p\eta, \quad u \to U\eta, \quad e \to 0, \quad \eta(t) = e^{\Lambda t}\eta(0), \tag{11}$$

for all initial state when $w = 0$ (see [6]).

To define an optimality criterion for output regulators, let

$$\zeta = C_o x_p + D_o u + F_o \eta \tag{12}$$

be the performance output. Then, from (11), we have

$$\zeta \to Z_o\eta, \quad Z_o := C_o X_p + D_o U + F_o, \tag{13}$$

where the steady state $Z_o\eta$ is not necessarily zero. The transient part of ζ is defined by

$$z := \zeta - Z_o\eta = C_o x_p + D_o u + (F_o - Z_o)\eta. \tag{14}$$

We consider minimization of the H_2 norm of the closed-loop transfer function from w to z over the set of controllers achieving (11) for all initial state when $w = 0$.

This optimal output regulation problem can be formulated as Problem 1 as follows. Note that plant (7), exosystem (8), and performance output (14) can be expressed as the generalized plant (2) with state $x := \mathrm{col}(x_p, \eta)$ and augmented matrices

$$A := \begin{bmatrix} A_o & E_o \\ 0 & \Lambda \end{bmatrix}, \quad B_1 := \begin{bmatrix} G_o \\ H_o \end{bmatrix}, \quad B_2 := \begin{bmatrix} B_o \\ 0 \end{bmatrix}.$$

$$C_1 := \begin{bmatrix} C_o & F_o - Z_o \end{bmatrix}, \quad D_1 := D_o. \tag{15}$$

When (X_p, U) satisfies (10), the regulator equation (5) is satisfied by $X := \mathrm{col}(X_p, I)$ because (5a) is identical to (10a), and (5b) holds by construction of z. The regulator equation for the error, (10b), is not explicitly included in the description of Problem 1 but it is implicit in the choice of (X, U).

C. Autonomous Pattern Generation Problem

Problem 1 also captures the autonomous pattern generation [13]. Consider the plant in (2) with an additional output

$$h = C_h x + D_h u.$$

Given a pair of matrices (H, Λ), the goal is to design a feedback controller $u = \mathbf{K}y$ such that every initial state response of the closed-loop system with $w = 0$ satisfies convergence to a prescribed pattern $h \to He^{\Lambda t}\eta_o$ for some η_o, without being driven by an exogenous system. Matrices H and Λ specify the steady state pattern in terms of the spatial (relative phases/amplitudes) and temporal (constant, oscillation) properties, respectively. See [10], [13], [15] for details.

Among the controllers achieving the goal, we may select the one that is optimal in the following sense. It has been shown [10], [13] that a requirement for feasibility of the goal is the existence of (X, U) satisfying

$$\begin{aligned} AX + B_2 U &= X\Lambda, \\ C_h X + D_h U &= H, \end{aligned} \tag{16}$$

and a feasible controller achieves convergence as in (6). Since $\mathrm{col}(x, u)$ converges to the range space of $\mathrm{col}(X, U)$, we may choose the performance output z to be the projection of $\mathrm{col}(x, u)$ onto the orthogonal complement of the range space so that fast convergence $z \to 0$ corresponds to fast convergence to the desired pattern. Thus we select

$$C_1 := \begin{bmatrix} X_\perp \\ -UX^- \end{bmatrix}, \quad D_1 := \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{17}$$

and minimize the H_2 norm of the closed-loop transfer function from w to z over the controllers achieving $h \to He^{\Lambda t}\eta_o$. This optimal pattern generation problem can be formulated as Problem 1 since (5b) is satisfied.

III. Main Results

This section presents the state/output feedback controllers that solve Problem 1. We will motivate the results with intuitive arguments in this section, followed by a rigorous proof of the main result in Section IV. Let us first consider the state feedback case. Under Assumption 1, using coordinate transformation (see e.g. [16] for similar developments)

$$\begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} := \begin{bmatrix} X_\perp \\ X^- \end{bmatrix} x, \quad \mu := u - Ux_{\bar{o}}, \tag{18}$$

the system (2) can be described by

$$\begin{bmatrix} \dot{x}_o \\ \dot{x}_{\bar{o}} \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{A}_o & 0 & \mathcal{B}_o \\ * & \Lambda & * \\ \mathcal{C}_o & 0 & \mathcal{D}_o \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \\ \mu \end{bmatrix} + \begin{bmatrix} * \\ * \\ 0 \end{bmatrix} w, \tag{19}$$

where

$$\begin{bmatrix} \mathcal{A}_o & \mathcal{B}_o \\ \mathcal{C}_o & \mathcal{D}_o \end{bmatrix} := \begin{bmatrix} X_\perp A X_\perp^- & X_\perp B_2 \\ C_1 X_\perp^- & D_1 \end{bmatrix}. \tag{20}$$

Consider minimization of $\|\mathcal{H}\|_2$ as specified in Problem 1 (s2). In the case of the standard LQR, due to the requirement of internal stability, the state $x_{\bar{o}}$ must be retained although it is unobservable from z. For Problem 1, however, we *may* remove this state since the Λ dynamics should be preserved for the closed-loop system to achieve (6). In this case, we would design the LQR of the form $\mu = \mathcal{K}_o x_o$ for the auxiliary plant $(\mathcal{A}_o, \mathcal{B}_o, \mathcal{C}_o, \mathcal{D}_o)$. It is well known that the LQR exists when

Assumption 4: $(\mathcal{A}_o, \mathcal{B}_o, \mathcal{C}_o, \mathcal{D}_o) \in \mathbb{S}$

holds. The corresponding controller for the original plant, $u = \mathcal{K}_o x_o + Ux_{\bar{o}}$, turns out to give the solution to Problem 1.

Theorem 1: *Consider the generalized plant (2), let matrices (X, U, Λ) be given, and define (20). Suppose Assumptions 1, 2, and 4 are satisfied, and $C_2 = I$ and $D_2 = 0$. Then an optimal* *static state feedback control that satisfies specifications (s1) and (s2) in Problem 1 is given by*

$$u = \mathcal{K}x, \quad \mathcal{K} := UX^- + \mathcal{K}_o X_\perp, \tag{21a}$$

$$\mathcal{K}_o := -(B_o^\intercal \mathcal{P}_o + D_o^\intercal C_o), \tag{21b}$$

where $\mathcal{P}_o$ is the stabilizing solution to the Riccati equation

$$\mathcal{P}_o A_o + A_o^\intercal \mathcal{P}_o + C_o^\intercal C_o = (\mathcal{P}_o B_o + C_o^\intercal D_o)(\mathcal{P}_o B_o + C_o^\intercal D_o)^\intercal.$$

The minimum H_2 norm achieved by this controller is

$$\|\mathcal{H}\|_2^2 = \mathrm{tr}(B_1^\intercal \mathcal{P} B_1), \quad \mathcal{P} := X_\perp^\intercal \mathcal{P}_o X_\perp. \tag{22}$$

Proof. With the coordinate transformation (18), the plant (2) is described by (19). In general, the static state feedback control is of the form

$$u = \mu + Ux_{\bar{o}}, \quad \mu = K_o x_o + K_{\bar{o}} x_{\bar{o}} \tag{23}$$

with some gains K_o and $K_{\bar{o}}$. Note that condition (6) in Problem 1 is equivalent to

$$x_o(t) \to 0, \quad x_{\bar{o}}(t) \to e^{\Lambda t}\eta_o, \quad \mu(t) \to 0. \tag{24}$$

We then see that specification (s1) is satisfied only if $K_{\bar{o}} = 0$. Therefore, the state $x_{\bar{o}}$ can be removed from (19) and the optimal control $\mu = \mathcal{K}_o x_o$ that minimizes $\|\mathcal{H}\|_2$ is given by the standard LQR theory applied to the system (A_o, B_o, C_o, D_o). The result then follows by noting that the controller $\mu = \mathcal{K}_o x_o$ achieves convergence of x_o and μ to zero for any initial condition under $w = 0$, and that the state $x_{\bar{o}}$ follows the dynamics $\dot{x}_{\bar{o}} = \Lambda x_{\bar{o}}$ in the steady state. ∎

Theorem 1 reveals the following architecture of the optimal state feedback control. The second term of $\mathcal{K}$ in (21) provides the feedback of the error $X_\perp x$ to achieve the convergence of x to the range space of X, i.e., $x_o \to 0$, while the first term is the feedforward of $Ue^{\Lambda t}\eta_o$ in the steady state to remain on the zero dynamics. Note that the auxiliary system (19) is obtained through the feedback transformation by the first term, $\mu = u - Ux_{\bar{o}}$, so that the eigenvalues of the zero dynamics (i.e., those of Λ) are eigenvalues of (19). The feedback control by the second term then preserves the Λ eigenvalues and stabilizes the remaining eigenvalues for the closed-loop system.

It is worth noting that $\mathcal{P}$ in (22) satisfies the Riccati equation for the original plant (2):

$$\mathcal{P}A + A^\intercal \mathcal{P} + C_1^\intercal C_1 = (\mathcal{P}B_2 + C_1^\intercal D_1)(\mathcal{P}B_2 + C_1^\intercal D_1)^\intercal, \tag{25}$$

and the additional properties

$$\begin{aligned} &\mathcal{P}X = 0, \quad \mathrm{eig}(A + B_2\mathcal{K}) \backslash \mathrm{eig}(\Lambda) \subset \mathbb{C}_-, \\ &\mathcal{K} := -(B_2^\intercal \mathcal{P} + D_1^\intercal C_1), \end{aligned} \tag{26}$$

where $\mathcal{K}$ in (21) and (26) are identical to each other. This can be easily verified by direct calculations to show the following identities

$$W^\intercal \Delta W = \begin{bmatrix} \Delta_o & 0 \\ 0 & 0 \end{bmatrix}, \quad W := \begin{bmatrix} X_\perp^- & X \end{bmatrix},$$

$$W^{-1}(A + B_2\mathcal{K})W = \begin{bmatrix} A_o + B_o\mathcal{K}_o & 0 \\ * & \Lambda \end{bmatrix},$$

where Δ and Δ_o are the left hand side minus the right hand side of the Riccati equations in (25) and Theorem 1, respectively. Among multiple solutions to the Riccati equation (25), we shall call $\mathcal{P}$ given by (22) the *partially stabilizing solution with respect to* (X, Λ) due to the properties in (26). It can readily be verified that

$$\begin{aligned} &(A + B_2\mathcal{K})X = X\Lambda, \\ &(C_1 + D_1\mathcal{K})X = 0, \end{aligned}$$

holds for $\mathcal{K}$ in (26) by noting that (21) implies $U = \mathcal{K}X$. Thus, the partially stabilizing solution provides a state feedback gain $\mathcal{K}$ that assigns the eigenstructure (X, Λ) to the closed-loop system. The recognition of Problem 1 as an eigenstructure assignment will be crucial for the proof of the next result.

We now generalize the result to the output feedback case. While it cannot be assumed that a separation principle holds for Problem 1, it turns out that it does and an optimal controller is given by replacing x in the state feedback described in Theorem 1 by the state estimate from the Kalman filter, which exists when

Assumption 5: $(A, B_1, C_2, D_2) \in \mathbb{S}^\intercal$

is satisfied. The result is formally stated as follows.

Theorem 2: *Consider the generalized plant (2), let matrices (X, U, Λ) be given, and define (20). Suppose Assumptions 1, 2, 4, and 5 are satisfied. Let $\mathcal{P}$ and $\mathcal{Q}$ be solutions of the Riccati equations*

$$\mathcal{P}A + A^\intercal \mathcal{P} + C_1^\intercal C_1 = (\mathcal{P}B_2 + C_1^\intercal D_1)(B_2^\intercal \mathcal{P} + D_1^\intercal C_1), \tag{27a}$$

$$A\mathcal{Q} + \mathcal{Q}A^\intercal + B_1 B_1^\intercal = (\mathcal{Q}C_2^\intercal + B_1 D_2^\intercal)(C_2\mathcal{Q} + D_2^\intercal B_1), \tag{27b}$$

where $\mathcal{Q}$ is the stabilizing solution and $\mathcal{P}$ is the partially stabilizing solution with respect to (X, Λ). Define the corresponding gains by

$$\mathcal{K} := -(B_2^\intercal \mathcal{P} + D_1^\intercal C_1), \quad \mathcal{F} := -(\mathcal{Q}C_2^\intercal + B_1 D_2^\intercal). \tag{28}$$

Then an optimal controller that solves Problem 1 is given by

$$\begin{aligned} &\dot{\hat{x}} = A\hat{x} + B_2 u + \mathcal{F}(C_2\hat{x} - y), \\ &u = \mathcal{K}\hat{x}, \end{aligned} \tag{29}$$

and the minimum H_2 norm is given by

$$\begin{aligned} \|\mathcal{H}\|_2^2 &= \mathrm{tr}(B_1^\intercal \mathcal{P} B_1) + \mathrm{tr}(\mathcal{K}\mathcal{Q}\mathcal{K}^\intercal) \\ &= \mathrm{tr}(C_1 \mathcal{Q} C_1^\intercal) + \mathrm{tr}(\mathcal{F}^\intercal \mathcal{P} \mathcal{F}). \end{aligned} \tag{30}$$

The formula for the optimal controller in Theorem 2 appears identical to the solution for the standard H_2 control problem [14], given in terms of the Kalman filter plus the linear quadratic regulator (LQR). However, since $\mathcal{P}$ is the partially stabilizing solution, the state feedback gain $\mathcal{K}$ is different from the LQR, and the closed-loop system is not internally stable. In particular, it is described by (18), $e := x - \hat{x}$, and

$$\begin{bmatrix} \dot{e} \\ \dot{x}_o \\ \dot{x}_{\bar{o}} \\ z \end{bmatrix} = \begin{bmatrix} A + \mathcal{F}C_2 & 0 & 0 & B_1 + \mathcal{F}D_2 \\ -B_o\mathcal{K} & A_o + B_o\mathcal{K}_o & 0 & X_\perp B_1 \\ * & * & \Lambda & X^- B_1 \\ -D_o\mathcal{K} & C_o + D_o\mathcal{K}_o & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ x_o \\ x_{\bar{o}} \\ w \end{bmatrix}.$$

Clearly, the closed-loop system contains the unstable modes associated with Λ, but the modes are unobservable and the transfer function $\mathcal{H}(s)$ from w to z is stable.

When (A, B_2) is stabilizable and Λ has no eigenvalues on the imaginary axis, the stabilizing solution $\mathcal{P} = \mathcal{P}_s$ to the Riccati equation (27a) exists. In this case, (29) gives the standard H_2 optimal controller, which internally stabilizes the closed-loop system and achieves $z(t) \to 0$ for any initial state under $w = 0$. The H_2 norm of the corresponding closed-loop transfer function is greater than or equal to that of $\mathcal{H}(s)$ achieved by (29) with the partially stabilizing solution $\mathcal{P}$ because $\mathcal{P}_s \geq \mathcal{P}$ due to maximality of the stabilizing solution [5], where the equality holds if Λ is Hurwitz. This means that the H_2 norm can possibly be made smaller by relaxing the stability requirement and allowing for convergence to the unstable zero dynamics when Λ has eigenvalues in the closed right half plane.

While the result in Theorem 2 is simple and makes sense intuitively, its proof is nontrivial. The main challenge is rigorous justification of the separation principle for Problem 1. As part of the proof, it should also be shown that the optimal state feedback is static. Theorem 1 shows optimality of (21) among the set of static state feedback controllers, but does not eliminate the possibility that a dynamic state feedback may give better performance (i.e., smaller $\|\mathcal{H}\|_2$). The direct generalization of Theorem 1 for the dynamic case is difficult mainly because, when (23) is replaced by

$$u = \mu + U x_{\bar{o}}, \quad \mu = K_o(s) x_o + K_{\bar{o}}(s) x_{\bar{o}},$$

specification (s1) no longer implies $K_{\bar{o}}(s) \equiv 0$. The facts that the separation principle holds and that the optimal state feedback is static are well known for the standard H_2 control problem. However, it is not clear if these properties hold for Problem 1 since it is non-traditional in that the closed-loop system is required to be internally unstable, making the state converge to a nonzero trajectory as in (6). In fact, it has been proven for a problem closely related to Problem 1 [13] that the optimal state feedback is dynamic, and the separation principle does not hold. We will show these two properties indeed hold, and prove Theorem 2 with the aid of the eigenstructure assignment theory [10] later in Section IV.

Let us now consider the optimal output regulation problem as a special case of Problem 1 for the generalized plant (2) with the specific structure (15) as described in Section II-B. Theorem 2 is specialized to reveal the optimal output regulation architecture as follows.

Corollary 1: *Consider the plant (7), exosystem (8), and performance output (12), and let matrices (X_p, U, Λ) be given. Define Z_o by (13) and system matrices by (15). Suppose Assumptions 2 through 5 are satisfied. Let $\mathcal{P}_o$ and $\mathcal{Q}$ be the stabilizing solutions of the Riccati equations*

$$\begin{aligned} &\mathcal{P}_o A_o + A_o^\intercal \mathcal{P}_o + C_o^\intercal C_o = (\mathcal{P}_o B_o + C_o^\intercal D_o)(B_o^\intercal \mathcal{P}_o + D_o^\intercal C_o), \\ &A\mathcal{Q} + \mathcal{Q}A^\intercal + B_1 B_1^\intercal = (\mathcal{Q}C_2^\intercal + B_1 D_2^\intercal)(C_2 \mathcal{Q} + D_2^\intercal B_1), \end{aligned}$$

and define the corresponding gains by

$$\mathcal{K}_o := -(B_o^\intercal \mathcal{P}_o + D_o^\intercal C_o), \quad \mathcal{F} := -(\mathcal{Q}C_2^\intercal + B_1 D_2^\intercal).$$

Then, the closed-loop system with the controller

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_2 u + \mathcal{F}(C_2 \hat{x} - y), \\ u &= U\hat{\eta} + \mathcal{K}_o(\hat{x}_p - X_p \hat{\eta}), \end{aligned} \quad \begin{bmatrix} \hat{x}_p \\ \hat{\eta} \end{bmatrix} := \hat{x} \tag{31}$$

satisfies the following convergence property: with an arbitrary initial state under $w = 0$, it holds that

$$\begin{aligned} &x_p \to X_p \eta, \quad u \to U\eta, \quad \eta = e^{\Lambda t}\eta(0), \\ &e \to 0, \qquad \zeta \to Z_o \eta. \end{aligned} \tag{32}$$

Moreover, this controller gives the minimum H_2 norm of the closed-loop transfer function $\mathcal{H}(s)$ from w to z among the set of all controllers satisfying the convergence property, where $z := \zeta - Z_o \eta$ is the transient part of ζ.

Proof. The result is a special case of Theorem 2 with the specific structure of the system matrices for the generalized plant (2) as in (15). We note that Assumption 3 implies Assumption 1 with $X := \mathrm{col}(X_p, I)$, and the matrices (A_o, B_o, C_o, D_o) in (20) with

$$\begin{bmatrix} X_\perp^- & X \end{bmatrix} = \left[\begin{array}{c|c} I & X_p \\ 0 & I \end{array}\right], \quad \begin{bmatrix} X_\perp \\ X^- \end{bmatrix} = \left[\begin{array}{cc} I & -X_p \\ \hline 0 & I \end{array}\right].$$

coincide with the corresponding plant matrices in (15). The result then follows from Theorem 2 using the expression of $\mathcal{K}$ in (21) and the relationship between $\mathcal{P}$ and $\mathcal{P}_o$ in (22), where the stated convergence property corresponds to (s1) in Problem 1 with $\eta_o = \eta(0)$. ∎

Corollary 1 shows that the optimal output regulator (31) has the following architecture. The steady state trajectories of x_p and u in (32) guarantee output regulation $e \to 0$ through the regulator equation (10), and are set as the design target. The plant state x_p and the output η from the exogenous system are estimated by the Kalman filter, and the control input u consists of the feedback and feedforward terms. The perturbation of x_p from the target trajectory $X_p \eta$ is estimated as $\hat{x}_p - X_p \hat{\eta}$, multiplied by the LQR gain $\mathcal{K}_o$, and used in the feedback term to stabilize the target trajectory. The feedforward term $U\hat{\eta}$ is the estimate of the persistent control input $u = U\eta$ needed to remain on the target trajectory.

IV. Proof of The Main Result

This section proves Theorem 2 within the framework of eigenstructure assignment. We will show that specification (s1) of Problem 1 can be formalized as an eigenstructure property (Lemma 1), parametrize feasible controllers (Lemma 2), reduce Problem 1 to the standard H_2 optimal control problem (Lemma 3), provide the state feedback solution (Lemma 4), and prove the output feedback result in Theorem 2. The core ideas for the first two lemmas are from [10], while the rest is newly developed here. To this end, let us first introduce:

Definition 1: A controller $u = \mathbf{K}y$ with a particular state space realization is said to be admissible if it is a detectable realization and satisfies specification (s1) of Problem 1, and the set of all admissible controllers is denoted by $\mathbb{A}$.

We first recognize that the control design with specification (s1) is an eigenstructure assignment.

Lemma 1: *Consider the plant (2), where Assumptions 1 and 2 are satisfied and D_1 has a full column rank. For each controller $u = \mathbf{K}y$ with a detectable realization, let the closed-loop system with $w = 0$ be described by $\dot{\mathsf{x}} = \mathsf{A}_{c\ell}\mathsf{x}$ with $\mathsf{x} = \mathrm{col}(x, x_c)$ where x_c is the controller state. Let $r := \mathrm{col}(x, u)$ be an output of the closed-loop system and define $\mathsf{H}_{c\ell}$ such that $r = \mathsf{H}_{c\ell}\mathsf{x}$. Then $\mathbf{K} \in \mathbb{A}$ holds if and only if*

$$\mathsf{A}_{c\ell}\mathcal{X} = \mathcal{X}\Lambda, \quad \mathsf{H}_{c\ell}\mathcal{X} = R, \quad \mathrm{eig}(\mathsf{A}_{c\ell}) \backslash \mathrm{eig}(\Lambda) \in \mathbb{C}_-, \tag{33}$$

hold for some matrix X_c, where

$$\mathcal{X} := \mathrm{col}(X, X_c), \quad R := \mathrm{col}(X, U).$$

In this case, the closed-loop transfer function $\mathcal{H}(s)$ is stable.

Proof. It can readily be verified using the PBH test that the pair $(\mathsf{H}_{c\ell}, \mathsf{A}_{c\ell})$ is detectable since the controller is a detectable realization. Also, the pair (R, Λ) is observable, where $R := \mathrm{col}(X, U)$, since X has a full column rank. Specification (s1) corresponds exactly to the notion of asymptotic equivalence between the systems $(\mathsf{H}_{c\ell}, \mathsf{A}_{c\ell})$ and (R, Λ). By the proof of Lemma 1 in [10], the asymptotic equivalence holds if and only if there exists a full column rank matrix $\mathcal{X}$ satisfying (33). The equality $\mathsf{H}_{c\ell}\mathcal{X} = R$ implies that $\mathcal{X}$ is of the form $\mathcal{X} = \mathrm{col}(X, X_c)$ for some X_c. This proves the equivalence. The stability of $\mathcal{H}(s)$ follows by observing that every impulse response satisfies (6) and therefore $z \to 0$ due to (5b). ∎

With an admissible controller $\mathbf{K} \in \mathbb{A}$, all the modes of the closed-loop system are stable except for unstable modes captured by Λ, and the output z converges to zero for an arbitrary initial state when $w = 0$. Lemma 1 reduces Problem 1 to an optimal eigenstructure assignment [10] to satisfy the conditions in (33) while minimizing the H_2 norm of the closed-loop transfer function $\mathcal{H}(s)$ from w to z. The control design is facilitated by a parametrization of a subset of $\mathbb{A}$ containing an optimal control, given as follows.

Lemma 2: *Consider the plant (2), where Assumptions 1 and 2 hold and (C_2, A) is detectable. The set of admissible controllers $\mathbb{A}$ is nonempty if and only if $(A, \bar{B}_2)$ is stabilizable, where $\bar{B}_2 := \mathrm{row}(B_2, -X)$. A subset of $\mathbb{A}$ is parametrized by detectable realizations of*

$$\begin{bmatrix} u \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} U \\ \Lambda \end{bmatrix} \xi + \mathbf{\Theta}(y - C_2 X \xi) \tag{34}$$

where $\mathbf{\Theta}$ is an arbitrary linear time-invariant state space system that internally stabilizes the augmented plant $(A, \bar{B}_2, C_2)$. Moreover, the optimal control that minimizes the H_2 norm of $\mathcal{H}(s)$ is an element of this subset.

Proof. The result essentially follows from Lemma 3, Theorem 3, and their proofs in [10]. (See also [13] that provides a brief proof for a version of this result). ∎

The PBH test verifies that stabilizability of $(A, \bar{B}_2)$ is equivalent to stabilizability of (A_o, B_o) in (20), which in turn implies existence of state feedback gain K such that the spectrum of $A + B_2 K$ contains the eigenvalues of Λ and all the other eigenvalues are in the open left half plane ($K = \mathcal{K}$ in (21a) in with $\mathcal{K}_o$ stabilizing $A_o + B_o\mathcal{K}_o$). When (A, B_2) is not stabilizable, addition of columns of X to B_2 may make $(A, \bar{B}_2)$ stabilizable. The implication for the original system (A, B_2) is the ability to assign the eigenstructure (X, Λ) and stabilize the other eigenvalues for the closed-loop system as in (26) so that convergence $x \to X e^{\Lambda t}\eta_o$ is achieved.

Exploiting the parametrization in (34), the following result converts Problem 1 to the traditional H_2 optimal control problem with internal stability of the closed-loop system.

Lemma 3: *Consider the plant $\mathbf{G}$ in (2), where Assumptions 1 and 2 hold and $(A, \bar{B}_2, C_2)$ is a stabilizable and detectable triple. Let a controller $\mathbf{K}$ be given by (34) with $\mathbf{\Theta}$ internally stabilizing $(A, \bar{B}_2, C_2)$. Define an augmented plant $\bar{\mathbf{G}}$ by*

$$\begin{array}{ll} \dot{\chi} = A\chi + B_1 w + \bar{B}_2 \mu, & \bar{B}_2 := \begin{bmatrix} B_2 & -X \end{bmatrix}, \\ z = C_1 \chi + \bar{D}_1 \mu, & \bar{D}_1 := \begin{bmatrix} D_1 & 0 \end{bmatrix}, \\ \varphi = C_2 \chi + D_2 w, & \end{array} \tag{35}$$

Then the closed-loop transfer function $\mathcal{H}(s)$ from w to z defined for $\mathbf{G}$ and $u = \mathbf{K}y$ coincides with the closed-loop transfer function $\bar{\mathcal{H}}(s)$ from w to z defined for $\bar{\mathbf{G}}$ and $\mu = \mathbf{\Theta}\varphi$. Moreover, the feedback system $(\mathbf{G}, \mathbf{K})$ is not internally stable, while the feedback system $(\bar{\mathbf{G}}, \mathbf{\Theta})$ is internally stable.

Proof. Consider the closed-loop system $(\mathbf{G}, \mathbf{K})$ with $\mathbf{K}$ described by (34). Label the input and output of $\mathbf{\Theta}$ as φ and μ, i.e., $\mu := \mathbf{\Theta}\varphi$. Pull out $\mathbf{\Theta}$ and define the rest of the system seen by $\mathbf{\Theta}$ as $\tilde{\mathbf{G}}$, which has input $\mathrm{col}(w, \mu)$, output $\mathrm{col}(z, \varphi)$, and state $\mathrm{col}(x, \xi)$ (this is a standard process in robust control [5]). Then the closed-loop system $(\mathbf{G}, \mathbf{K})$ is described as feedback system $(\tilde{\mathbf{G}}, \mathbf{\Theta})$. Using

$$\chi := x - X\xi, \quad \varphi := y - C_2 X \xi,$$

express $\tilde{\mathbf{G}}$ in terms of the state $\mathrm{col}(\chi, \xi)$ and simplify the equations using (5) for the zero dynamics. This realization of $\tilde{\mathbf{G}}$ is given by (35) and $\dot{\xi} = \Lambda\xi + \mathrm{row}(0, I)\mu$. The Λ modes are unobservable from $\mathrm{col}(z, \varphi)$, and ξ can be removed to obtain $\bar{\mathbf{G}}$ in (35). The feedback system $(\mathbf{G}, \mathbf{K})$ shares the eigenvalues with Λ and hence is not stable. The feedback system $(\bar{\mathbf{G}}, \mathbf{\Theta})$ is internally stable because $\mathbf{\Theta}$ stabilizes $(A, \bar{B}_2, C_2)$. ∎

For solving Problem 1, the convergence property in (s1) is enforced by the controller structure in (34), and the optimality in (s2) can be achieved by choosing $\mathbf{\Theta}$ to optimize the H_2 norm of $\bar{\mathcal{H}}(s)$. The latter problem is a singular optimal control problem with rank-deficient $\bar{D}_1$ and may be addressed by classical results (e.g. [17]). However, the singularity is caused by embedding of the zero dynamics in the controller (34) and it is possible to obtain a simple and clean solution by exploiting the special structure of $\bar{\mathbf{G}}$. Here is the state feedback result.

Lemma 4: *Let the generalized plant (2) be given, where Assumptions 1, 2, and 4 are satisfied, and $C_2 = I$ and $D_2 = 0$. For the augmented plant (35) and possibly dynamic controller $\mu = \mathbf{\Theta}\chi$, let $\bar{\mathcal{H}}(s)$ be the closed-loop transfer function from w to z. Then, an optimal $\mathbf{\Theta}$ that internally stabilizes the closed-loop system with the smallest $\|\bar{\mathcal{H}}\|_2$ is a static gain given by*

$$\mu = \bar{\mathcal{K}}\chi, \quad \bar{\mathcal{K}} := \mathrm{col}(\mathcal{K}, EX^-)$$

where $\mathcal{K}$ is defined by (21) and E is an arbitrary matrix such that $\Lambda - E$ is Hurwitz. Moreover, the static controller in (21) is the minimal realization of (34) with the optimal $\mathbf{\Theta}$, and solves Problem 1 with optimality in (s2) among all static and dynamic controllers satisfying (s1).

Proof. Direct substitution verifies that the controller (21) can be expressed as (34) with system $\mu = \mathbf{\Theta}\varphi$ given by constant gain $\mu = \bar{\mathcal{K}}\varphi$, where ξ is unobservable in u and can be removed. We show that $\mu = \bar{\mathcal{K}}\chi$ is an optimal state feedback that minimizes $\|\bar{\mathcal{H}}\|_2$. First note that $\bar{\mathcal{K}}$ stabilizes the augmented plant (35) since

$$W^{-1}(A + \bar{B}_2\bar{\mathcal{K}})W = \begin{bmatrix} A + B_o\mathcal{K}_o & 0 \\ * & \Lambda - E \end{bmatrix},$$

where W and (A_o, B_o) are defined in (1) and (20). It is well known [18] that the optimal state feedback for the standard H_2 control problem is given by a static gain. Hence, it suffices to prove that $\mu = \bar{\mathcal{K}}\chi$ gives the smallest H_2 norm achievable by any static state feedback with internal stability. Let $\bar{K}$ be an arbitrary matrix such that $A + \bar{B}_2\bar{K}$ is Hurwitz.

From the standard linear system theory, the H_2 norm squared of the closed-loop transfer function $\bar{\mathcal{H}}(s)$ for the controller $\mu = \bar{K}\chi$ is given by $\mathrm{tr}(B_1^\intercal P B_1)$ where P is the solution to the Lyapunov equation

$$P(A+\bar{B}_2\bar{K})+(A+\bar{B}_2\bar{K})^\intercal P+(C_1+\bar{D}_1\bar{K})^\intercal(C_1+\bar{D}_1\bar{K}) = 0.$$

Note that $P = \mathcal{P}$ is the solution when $\bar{K} = \bar{\mathcal{K}}$. We complete the proof by showing that $P \geq \mathcal{P}$ holds for any $\bar{K}$ and hence we have $\mathrm{tr}(B_1^\intercal P B_1) \geq \mathrm{tr}(B_1^\intercal \mathcal{P} B_1)$, implying that $\mu = \bar{\mathcal{K}}\chi$ gives the smallest H_2 cost. To see this, subtract the Riccati equation (25) from the above Lyapunov equation to get

$$\begin{aligned} &P_\Delta(A+\bar{B}_2\bar{K}) + (A+\bar{B}_2\bar{K})^\intercal P_\Delta + K_\Delta^\intercal \bar{D}_1^\intercal \bar{D}_1 K_\Delta = 0, \\ &P_\Delta := P - \mathcal{P}, \quad K_\Delta := K_* - \bar{K}, \quad K_* := \mathrm{col}(\mathcal{K}, 0). \end{aligned}$$

This implies $P \geq \mathcal{P}$ when $A + \bar{B}_2\bar{K}$ is stable because P_Δ is an integral of the positive semidefinite forcing term. ■

Lemma 4 shows that the solution $\bar{\mathcal{K}}$ to the singular optimal control for (35) is characterized by the partially stabilizing Riccati solution $\mathcal{P}$, and is not unique due to the freedom in E. This freedom does not affect the optimal controller (21) for the original plant since u in (34) is independent of E. There are two roles of Lemma 4. One is to show that the static state feedback (21) is indeed the optimal solution to Problem 1 and dynamic state feedback does not help to improve the $\|\mathcal{H}\|_2$ performance. The other role is to set a basis for the proof of the general output feedback solution presented next.

Proof of Theorem 2. Based on the separation principle (Theorem 4.1 of [18]), the optimal control $\mu = \boldsymbol{\Theta}\varphi$ for the augmented plant (35) to minimize the H_2 norm of $\bar{\mathcal{H}}(s)$ is given by the optimal state feedback plus the Kalman filter:

$$\mu = \bar{\mathcal{K}}\hat{\chi}, \quad \dot{\hat{\chi}} = A\hat{\chi} + \bar{B}_2\mu + \mathcal{F}(C_2\hat{\chi} - \varphi).$$

By Lemmas 2 and 3, an optimal control solving Problem 1 is given by substituting this $\boldsymbol{\Theta}$ into (34), resulting in

$$\begin{aligned} u &= U\xi + \mathcal{K}\hat{\chi}, \\ \dot{\xi} &= \Lambda\xi + EX^-\hat{\chi}, \\ \dot{\hat{\chi}} &= (A+\bar{B}_2\bar{\mathcal{K}})\hat{\chi} + \mathcal{F}(C_2\hat{\chi} - y + C_2X\xi). \end{aligned}$$

Introducing the new state coordinate

$$\hat{x} := \hat{\chi} + X\xi,$$

this controller is described by (29), where we used $U = \mathcal{K}X$ and the regulator equation (5), and noted that the state ξ is unobservable from u and can be eliminated. The minimum H_2 norm in (30) follows from a formula in [18] and its dual, applied to the augmented closed-loop system $\bar{\mathcal{H}}(s)$. ■

V. Design Example

Consider a mass-spring system subject to actuator force f and disturbance force η, which is described by $m\ddot{p} = f+\eta-kp$ where m and k are the mass and stiffness, and p is the position. The force f is generated by the control input u through the actuator dynamics $f + \tau\dot{f} = u$, where τ is the time constant. With $m = \tau = 1$ and $k = 4$, the system is described by the first equation in (7) with $x_p := \mathrm{col}(p, \dot{p}, f)$, $G_o = 0$, and

$$A_o = \begin{bmatrix} 0 & 1 & 0 \\ -k & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad B_o = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad E_o = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The measured output is $y = p + w_v$, where w_v is the sensor noise. The force η is modeled by (8) with $\Lambda = \Lambda_d$ and

$$\Lambda_d = 0, \quad H_o = \begin{bmatrix} \alpha & 0 \end{bmatrix}, \quad w := \mathrm{col}(w_d, w_v),$$

where α is a design weight representing the magnitude of the disturbance, which is a constant when w_d is an impulse.

We will design a controller $u = \mathbf{K}y$ such that the position or velocity converges to zero under the constant disturbance. The design will minimize the L_2 norm of the transient part of $\zeta := \mathrm{col}(\beta p, u)$, where β is a design weight. The design can be formulated as Problem 1 with (15) and solved by Theorem 2. The solution to the regulator equation (10a) is given by

$$X_p = \mathrm{col}(\rho, 0, k\rho - 1), \quad U = k\rho - 1,$$

where ρ is an arbitrary parameter. If $\rho = 0$, then the position p is regulated at zero. If $\rho \neq 0$, then the velocity $\dot{p}$ is regulated at zero with the position p and control input u converging to $\alpha\rho$ and $\alpha(k\rho - 1)$, respectively, to maintain the force balance in the steady state. The design result is shown in Fig. 1, where the impulse responses under $w_d = \delta(t)$ and $w_v = 0$ are shown for three cases $\rho = 0, 1, 2$ (blue, red, yellow) with the design weights $\alpha = 2$, $\beta = 5$. When $\rho = 0$, the controller has an integrator as an internal model and makes p converge to zero while u balances $\eta = 2$ in the steady state. We see that the choice of ρ in X_p specifies the steady state position.

Next, we design a controller $u = \mathbf{K}y$ such that the position $h := p$ oscillates with frequency ω in the steady state in the presence of the persistent constant disturbance. Choose

$$H_\pi := \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \Lambda_\pi = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix},$$

and consider the target behavior $h \to H_\pi e^{\Lambda_\pi t}\eta_\pi$ for some η_π. Then, for the augmented plant (15) with $\Lambda := \Lambda_d$, the solution to the regulator equations (16) with $\Lambda := \Lambda_\pi$ and $H := H_\pi$ is given by $(X, U) = (X_\pi, U_\pi)$ where

$$X_\pi = \begin{bmatrix} 1 & 0 \\ 0 & \omega \\ k_\omega & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{aligned} &U_\pi = k_\omega \begin{bmatrix} 1 & \omega \end{bmatrix}, \\ &k_\omega := k - \omega^2. \end{aligned}$$

The choice $X := X_\pi$ leads to not-stabilizable (A_o, B_o) in (20), violating Assumption 4 and making Problem 1 infeasible. This is because A contains uncontrollable mode Λ_d, which has to remain in the closed-loop system. Therefore, we redefine $X := \mathrm{row}(X_\pi, X_d)$ and $U := \mathrm{row}(U_\pi, U_d)$, where (X_d, U_d) is equal to (X, U) in the previous design, i.e., $X_d = \mathrm{col}(X_p, 1)$ and $U_d = k\rho - 1$. In this case, (X, U) satisfies (16) with $\Lambda := \mathrm{diag}(\Lambda_\pi, \Lambda_d)$ and $H := \mathrm{row}(H_\pi, \rho)$, and the target behavior is modified as $h \to He^{\Lambda_\pi t}\eta_\pi + \alpha\rho$.

Using Theorem 2, the optimal controller is designed for the cost defined by (17) with $X_\perp$ replaced by $\beta X_\perp$ with normalization $X_\perp X_\perp^\intercal = I$. The design result is shown in Fig. 1 for two cases $(\rho, \omega) = (1, 1)$ and $(1, 2)$ with $\alpha = 2$ and $\beta = 5$. When $\omega = 1$, both p and u converge to sinusoids with period 2π, where the average values coincide with the steady state values in the previous output regulation design with $\rho = 1$. When $\omega = 2$, which is equal to the natural frequency $\sqrt{k/m}$ of the mechanical system, u converges to a constant that balances with the disturbance and average spring force, while p converges to the natural oscillation.

These design problems cannot be solved by existing methods; not by [12] due to the presence of sensor noise w_v, and not by [13] due to the lack of (A, B_2) stabilizability.

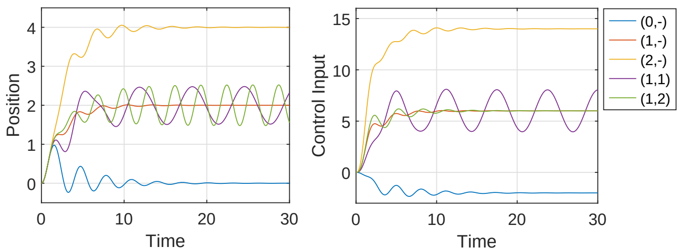

Fig. 1. Time responses. The legend indicates (ρ, ω).

VI. Discussion and Conclusion

We have considered an optimal regulation problem to achieve convergence to the zero dynamics (Section II-A), which unifies the classical output regulation with exosystem (Section II-B) and the autonomous pattern generation without exosystem (Section II-C). The problem is solved within the general framework of the eigenstructure assignment (Section IV), and the optimal control is characterized using the partially stabilizing solution to the Riccati equation (Section III), which arises due to the requirement that the closed-loop system embed eigenvalues of the anti-stable zero dynamics.

Theorem 2 solves a version of the optimal eigenstructure assignment, while another closely related version has recently been solved in [13]. They both solve Problem 1, but in different settings in terms of the cost function and the plant class. They compare as follows.

A general cost function is considered in [13], where the performance output ζ can be arbitrary and its transient part z is penalized. In our formulation, this is also the case when the output regulation is considered (see Section II-B), but is not the case in general since the transient part z of an arbitrary plant output ζ cannot always be defined as a linear combination of x and u. However, convergence to the subspace spanned by $\mathrm{col}(X, U)$ can be optimized (Section II-C), which is effective as demonstrated by a design example (Section V).

Our result has provided a new insight into the optimal control architecture. In particular, Theorem 2 revealed that, when the performance signal z is associated with the zero dynamics, the optimal controller does not explicitly contain an internal model of the zero dynamics Λ and the controller order is equal to the plant order n. This is in contrast with the optimal control with a general cost function in [13], which explicitly embeds an internal model of $\Lambda \in \mathbb{R}^{\ell \times \ell}$ and the order of the controller is $n + \ell$.

Finally, the class of plants is larger in Theorem 2 than in [13], where the latter assumes stabilizability of (A, B_2) and excludes the output regulation case with an unstable exosystem; see (15). Our result assumes stabilizability of (A_o, B_o) instead, which is a weaker condition associated with convergence to the range space of X, governed by the zero dynamics. This allows for integration of the output regulation and pattern generation to give more design flexibility, as illustrated by a design example in Section V.

For the optimal output regulation problem, our result compares with the state-of-the-art result in [12] as follows. The problem formulated in Section II-B is more general in the sense that the performance output ζ can be different from the regulated output e and does not have to converge to zero, and exogenous signals with unknown (unmodeled) dynamics can represent multiple channels of plant disturbances and sensor noises. Moreover, unlike the singular optimal control problem in [12], regularity conditions (full control penalty in ζ and no noise-free measurements in y) lead to the optimal controllers characterized by standard Riccati equations. This feature reveals an optimal control architecture comprising the Kalman filter, the LQR, and a disturbance/reference feedforward as in Corollary 1. While this architecture may be crafted from intuition, its optimality has not been previously proven in the literature.

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