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Striated Regularity of Vorticity in a Bounded Domain

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Striated Regularity of Vorticity in a Bounded Domain

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To Shahla

ABSTRACT OF THE DISSERTATION

Striated Regularity of Vorticity in a Bounded Domain

by

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Dr. James P. Kelliher, Chairperson

The two-dimensional Euler equations describe the velocity of an inviscid incompressible fluid. A classical vortex patch is a solution to the two-dimensional Euler equations whose initial vorticity is the indicator function of a bounded simply connected open region in the plane. Properties of the flow maps and vorticity transport in two dimensions ensure that the vorticity at any time will be the indicator function of the the image of the region, which remains simply connected and bounded. In 1991, Chemin proved in [Che91] that, in the whole plane, a vortex patch with an initially Hölder continuous boundary maintains that boundary regularity for all time. A few years later, Serfati published an alternate strategy in [Ser94b] that simplifies certain aspects of Chemin's argument. Here, we prove that, for $0 < \alpha < 1$, $C^{1,\alpha}$ regularity of a vortex patch boundary persists for all time for fluids in a simply connected bounded domain that itself has a smooth boundary, as long as the patch is initially not touching the boundary. The proof reproduces a 1998 result of Depauw ([Dep98]) using simpler methods inspired by Serfati's approach, which is more easily adaptable to a bounded domain than the methods of Chemin and Depauw.

Contents

Chapter 1

Introduction

1.1 The Two-Dimensional Euler Equations

The two-dimensional Euler equations describe the motion of an inviscid incompressible fluid in the whole plane. If $u(t, x)$ and $p(t, x)$ represent the fluid velocity and pressure, respectively, at time $t \geq 0$ and position $x \in \mathbb{R}^2$, then the motion satisfies the system of equations

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, & (1.1) \\ \operatorname{div} u = 0, & (1.2) \\ u(0, \cdot) = u_0. & (1.3) \end{cases}$$

A (classical) vortex patch is a solution (u, p) to this system whose initial vorticity $\omega_0 :=$ $\operatorname{curl} u_0$ is the indicator function of an open region U in the plane; that is, $\omega_0(x) = \mathbb{1}_U(x)$. In a slight abuse of terminology, we also refer to the region U as "the initial vortex patch."

In terms of the vorticity-stream formulation of the equations (Proposition 2.1 of [MB02]), the initial value problem in the full plane can be written as

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, & (1.4) \\ \omega(0, \cdot) = \omega_0 := \mathbb{1}_U, & (1.5) \\ u(t, \cdot) = [K * \omega](t, \cdot). & (1.6) \end{cases}$$

Here, (1.4) is obtained by taking the curl of the momentum conservation equation (1.1), (1.5) is the initial condition, and (1.6) expresses that the velocity $u(t, x)$ can be recovered from the vorticity $\omega(t, x)$ via the Biot-Savart law, where K is the Biot-Savart kernel in $\mathbb{R}^2$. The Biot-Savart law will be discussed in detail in Section 2.5.

Associated to the fluid motion is a flow map, which we denote by $\eta(t, x)$, that describes the particle trajectories for each fixed value of x. If x represents the position of a fluid particle at time $t = 0$, then $\eta(t, x)$ is the position of that particle after flowing in the fluid velocity field for time t. The flow map can be defined as the solution to the ordinary differential equation

$$\begin{cases} \partial_t \eta(t, x) = u(t, \eta(t, x)), \\ \eta(0, x) = x. \end{cases} \tag{1.7}$$

Since (1.4) dictates that the vorticity is passively transported and since the flow maps are continuous (see Chapter 2 for more details), an initially connected vortex patch will remain a connected vortex patch as time increases. Though the divergence-free condition (1.2) ensures that the patch will have constant area for all time, some of its geometric properties, such as the length and curvature of its boundary, may grow rapidly as the patch deforms over time.

Vortex patches have been studied going back to the 19th century. The most well-known example of a non-trivial vortex patch is the Kirchoff ellipse, first described in 1876 in [Kir76]. In this case, the vortex patch is initially a regular ellipse in the plane. As time increases, the ellipse maintains its shape but rotates at a constant angular velocity depending on the relative lengths of the axes of the ellipse.

1.2 The Vortex Patch Problem in the Plane

Unique weak solutions to the vortex patch initial value problem described by (1.4)-(1.6) have been known to exist since Yudovich's 1963 work [Yud63], since $\omega_0 = \mathbb{1}_U$ is in $L^1 \cap L^\infty(\mathbb{R}^2)$. Weak solutions will be discussed in Section 2.4. A problem of much interest throughout the 1980s and early 1990s was: if the boundary of the vortex patch is initially smooth, does it stay smooth for all time?

Not much progress was made in this direction until 1979, when Zabusky, Hughes, and Roberts derived the contour dynamics equation (CDE) in [ZHR79]. Their idea was that, since the vortex patch is completely determined by its boundary and because points on the boundary must remain on the boundary as they flow, the evolution of the patch could be described by tracking the boundary alone; the CDE parametrizes the boundary of the vortex patch at time t. Using the notation of Section 8.3 of [MB02], if such a parametrization is represented by $z(t, \alpha) = \eta(t, z(0, \alpha))$, so that a parametrization of the intial patch boundary is transported by the flow, and if the boundary is at least piecewise

C^1, the motion of the boundary of the patch satisfies the CDE

$$\begin{cases} \dfrac{dz}{dt}(t,\alpha) = -\dfrac{\omega_0}{2\pi}\displaystyle\int_0^{2\pi} \ln\left|z(t,\alpha) - z(t,\alpha')\right| \partial_\alpha z(t,\alpha')\, d\alpha', \\ z(t,\alpha)|_{t=0} = z_0(\alpha). \end{cases} \tag{1.8}$$

This reduction in complexity from two dimensions to one dimension allowed computer models and simulations to come into play. In the 1980s, vortex patch dynamics became an active area of research in computational fluid dynamics with heated competition to come up with the most efficient and accurate algorithms to model the vortex patch evolution and provide evidence of either the persistence of the patch boundary's regularity or of the formation of singularities on the boundary.

In 1986, Majda conjectured in [Maj86], based on a simplified mathematical model and the contour dynamics equation, that the boundary of a vortex patch can in fact lose smoothness in finite time. He and Constantin had shown in [CLM85] that solutions to a certain scalar equation involving the Hilbert transform could blow up in finite time. His conjecture was based on analytic similarities he had observed between these solutions and solutions to the CDE. Majda's conjecture was supported by the 1989 computational results of Buttke in [But89], which were obtained using Buttke's *fast adaptive vortex method* developed in [But90]. His calculations showed evidence of the formation of a sharp corner on the boundaries of two initially identical circular vortex patches separated by half their common radius in finite time. This result was questioned later in 1989 by Dritschel and McIntyre in [DM90], where they suggested that Buttke's results were an artifact of his method's use of square-shaped spatial elements to approximate the patches and provided their own computational evidence that the boundaries actually do stay smooth well past the time at which singularities were claimed to emerge. This matter was debated for several more years before a mathematical proof finally came out.

1.3 The Strategies of Chemin and Bertozzi & Constantin

The question of whether a smooth vortex patch boundary in the whole plane remains smooth for all time was answered in the affirmative in the early 1990s by Chemin in [Che91, Che93]. This was also proved independently in [BC93] by Bertozzi and Constantin.

Bertozzi proved in her doctoral dissertation ([Ber91]) that there exist local-in-time $C^{1,\gamma}$ $(0 < \gamma < 1)$ solutions to the CDE, proving that Hölder regularity of vortex patch boundaries is maintained for some finite time, and gave sufficient conditions for such a solution to be continued for all time. In [BC93], Bertozzi and Constantin expanded that work and published a proof of global-in-time existence of $C^{1,\gamma}$ solutions to the CDE [1]. While their proof also affirmatively answered the vortex patch boundary regularity question, their strategy was less generalizable than Chemin's due to their reliance on the CDE. However, the approaches had a unifying principle behind them: in both cases, the regularity of the boundary was measured by a passively-transported function (or family of functions) that possessed higher regularity than the vorticity itself.

We will now briefly describe both approaches. Before discussing Chemin's strategy, it can be helpful to first examine what Bertozzi and Constantin did since their framework is that of a classical vortex patch and so may help build intuition for the more general problem that Chemin addressed. Bertozzi and Constantin's strategy was to link the regularity of the

[1] Bertozzi also showed in [Ber91] how the result for $C^{1,\gamma}$ boundaries naturally extends to $C^{k,\gamma}$ boundaries for any positive integer k, and, by induction, for $C^\infty = \cap_{k=1}^\infty C^{k,\gamma}$ boundaries.

vortex patch boundary to several quantities associated with the CDE (1.8) developed by Zabusky, Hughes, and Roberts in [ZHR79] that parametrized the boundary of the patch at any time t. They then proved global-in-time bounds on these quantities in terms of the initial data through standard singular integral estimation techniques combined with a technical geometric argument that was used to bound the most difficult piece of a certain integral. By representing the boundary of the vortex patch U_0 at time $t = 0$ by $\partial U_0 = \varphi_0^{-1}(0)$, where φ_0 is a $C^{1,\alpha}$ ($\alpha \in (0,1)$) scalar function, the boundary of the vortex patch ∂U_t at any time t could then be represented by letting φ_0 be passively transported by the flow: $\partial U_t = \varphi_t^{-1}(0)$, where $\varphi_t(\eta(t,x)) := \varphi_0(x)$ and η is the flow map for the velocity u defined by (1.7).

The quantities estimated were $\|\nabla u\|_{L^\infty}$, $\|\nabla \varphi_t\|_{L^\infty}$, $\|\nabla \varphi_t\|_{\dot{C}^\alpha}$, and $\|\nabla \varphi_t\|_{\text{inf}}$. The measurements being used here are the α-Hölder seminorm

$$\|f\|_{\dot{C}^\alpha} := \sup_{x \neq y} \frac{f(x) - f(y)}{|x-y|^\alpha}$$

and the *boundary infimum* $\|\nabla \varphi_t\|_{\text{inf}}$ defined as the infimum of $\nabla \varphi_t$ along the boundary ∂U_t of the vortex patch at time t. They obtained an initial estimate of $\|\nabla u\|_{L^\infty}$ from the Biot-Savart Law (Lemma 2.5.1) and then a chain of subsequent estimates was eventually closed with Grönwall's Lemma. The fact that these bounds give global regularity of the vortex patch boundary follows from standard ordinary differential equation theory; the details can be found, for instance, in Chapter 8 of [MB02].

A fundamental idea behind Bertozzi and Constantin's approach is that the regularity of the boundary is encapsulated by various measurements of the vector field $\nabla^\perp \varphi_t$, where the perpendicular gradient operator is defined as $\nabla^\perp := (-\partial_2, \partial_1)$. Because the vortex patch boundary at time t is a level set of φ_t, the vector field $\nabla^\perp \varphi_t$ is tangential to the boundary of the patch so it can be used to describe the manner in which the boundary is deforming as the fluid flows over time. Once one shows that $\varphi_t \in C^{1,\alpha}$, its perpendicular gradient $\nabla^\perp \varphi_t \in C^\alpha$ and so has more smoothness than the vorticity ω, which is only in L^∞. It is this ability to describe the vortex patch boundary using a higher-regularity tool than the vorticity itself that makes the proof possible.

Chemin proved his more general result (in 2D, but later extended to higher dimensions by Danchin in [Dan99]) by showing that certain quantities related to a *sufficient family* of vector fields $\mathcal{Y}$ remain bounded for all time. Sufficient families will be discussed in more detail below in (1.9). Chemin proved that if the initial vorticity is integrable and essentially bounded, and if the product of the sufficient family and the initial vorticity has its divergence in a negative Hölder space ($C^{\alpha-1}$, defined in Section 2.1), then the divergence of the product remains in $C^{\alpha-1}$ for all time. If you apply Chemin's result to the perpendicular gradient of the scalar functions φ_t that define the boundary of a vortex patch as in Bertozzi and Constantin's approach, then the bounds obtained in [BC93] follow, solving the classical vortex patch problem. However, Chemin's result can also be applied to more general problems such as patches of non-constant vorticity and even arbitrary level sets of any initial vorticity. While his proof relies heavily on the powerful machinery of paradifferential calculus, the essence of his strategy also comes down to proving global-in-time bounds on quantities mathematically similar to those used by Bertozzi and Constantin.

Roughly speaking, Chemin's sufficient family $\mathcal{Y}$ consists of vector fields with C^α-regularity in the tangential direction to level sets of the vorticity. The quantities he bounded were $\|\nabla u\|_{L^\infty}$, $\|\mathcal{Y}\|_{C^\alpha}$, and $\|\mathcal{Y} \cdot \nabla\omega\|_{C^\alpha}$ (see Section 2.1 for an explanation of the sufficient family notation)[2]. Recall that the norm $\|\cdot\|_{C^\alpha}$ is defined as the sum of the L^∞ norm and the α-Hölder seminorm. As with Bertozzi and Constantin's proof, the bounds were obtained with a series of estimates that were closed with Grönwall's Lemma. The difference lies in the tools used to obtain the bounds. Chemin was able to obtain a more general result because his methods did not rely on the contour dynamics equation.

1.4 Serfati's Strategy

In 1994, Serfati published another approach in [Ser94b]. Much of his short proof broadly mirrored the ideas of Chemin's 1993 paper [Che93], but his approach was markedly different in how it obtained a key estimate on $\|\nabla u\|_{L^\infty}$ that was used to close the series of estimates. This estimate was obtained by making clever use of a linear algebra lemma, presented in various forms in [Ser92, Ser94a, Ser94b], that bounds the Euclidean norm of a symmetric matrix in terms of its trace and an arbitrary symmetric matrix. In Section 3.7, we give the form of the lemma that will be needed as Lemma 3.7.1, which is based on the more general Lemma 5.1 of [BK15].

The various approaches of Chemin, Bertozzi and Constantin, and Serfati have some unifying principles underlying them. The core idea of each strategy is to first prove various transport estimates for the quantities of interest: for Chemin's approach, the sufficient family of vector fields $\mathcal{Y}$; for Bertozzi and Constantin's approach, $\nabla^\perp\varphi$; and for Serfati's approach, a single[3] vector field Y. The second step is to prove key estimates on $\|\nabla u(t,\cdot)\|_{L^\infty}$.

[2] Writing Chemin's assumption as $\mathcal{Y}\cdot\nabla\omega \in C^\alpha$ is slightly misleading since we only assume $\omega \in L^\infty$. Because Chemin also assumed that $\operatorname{div}\mathcal{Y} = 0$ in [Che91], it can be interpreted as

$$\mathcal{Y}\cdot\nabla\omega = \operatorname{div}(\omega\mathcal{Y}) - \omega\operatorname{div}\mathcal{Y} = \operatorname{div}(\omega\mathcal{Y}).$$

The third step is to close the chain of estimates with Grönwall's Lemma (Lemma 2.2.7). It is in the second step that the approaches diverge and where the primary difficulties lie. Indeed, Bertozzi and Constantin state in [BC93] that the entire result is "mainly due" to the properties (their so-called "kinematic reasons") of $\|\nabla u(t,\cdot)\|_{L^\infty}$. Chemin used the tools of paradifferential calculus to obtain his bounds while Bertozzi and Constantin used a geometric argument about the Lebesgue measure of a radial set used in a key integral that they called the Geometric Lemma in [BC93]; Serfati obtained the necessary bound using his linear algebra lemma.

Recently, Bae and Kelliher adopted and amplified the results of Chemin in [BK15] and [BK21] using Serfati's approach to obtain the critical estimates on $\|\nabla u\|_{L^\infty}$. They also adopted some of Danchin's ideas from [Dan99] to extend the results to higher dimensions. The proof presented here is an adaptation of their work to the 2D bounded domain setting. By *domain*, we mean a connected open set.

1.5 The Vortex Patch Problem in a Bounded Domain

The Euler equations can be studied in a bounded domain by the addition of the boundary condition $u \cdot \hat{n} = 0$, where $\hat{n}$ is the outward unit normal vector on the boundary of the domain. This merely states that no fluid flows in or out of the domain through the boundary. Yudovich's seminal work [Yud63] (see also section 8.2 of [MB02]) proved that there exist unique weak solutions (see Definition 2.4.1) to the vorticity-stream formulation

[3] Serfati's use of a single vector field limited his result's applicability. The vector field was meant to track the areas along the patch boundary where the vorticity was discontinuous, similar to $\nabla^\perp\varphi$ from [BC93], but if the vorticity was more complicated than a vortex patch, such as "layers" of vorticity, then Serfati's result could not be applied while Chemin's could. In the end, the main result presented here follows Serfati's approach but uses Chemin's sufficient families.

of the two-dimensional Euler equations in a bounded domain if the domain has a C^∞ boundary and the initial data $\omega_0 \in L^\infty$. In [GVL13], Gérard-Varet and Lecave showed that the smoothness assumption on the boundary of the domain could be weakened to $C^{1,1}$. However, these existence results do not address the regularity of a vortex patch boundary. The presence of a domain boundary complicates the vortex patch problem, mostly because of the fact that the Biot-Savart kernel in a bounded domain has an extra term that, while well-behaved in the interior of the domain, is singular along its boundary.

Several results have shown that under certain conditions the vorticity can become irregular in the presence of a boundary. In [KZ14], Kiselev and Zlatos provided an example of a bounded domain, smooth except at two interior cusps, on which there exist smooth classical solutions to the two-dimensional Euler equations that blowup in finite time in the sense that the vorticity loses continuity. In [CJM13], Crosby, Johnson, and Morrison used numerical techniques to investigate the behavior of vortex patches in the presence of various types of boundaries and provided evidence that singularities can form if the domain's boundary has a corner, such as a square domain. However, using methods similar to Chemin's approach, Depauw proved in [Dep98] the persistence of boundary regularity of a vortex patch in a bounded simply connected domain with C^∞ boundary as long the initial vortex patch is not touching the boundary. We note that Depauw was later able to obtain local-in-time boundary regularity for vortex patches that are initially tangent to the boundary of the domain in [Dep99], but not global-in-time regularity.

Our main theorem, Theorem 1.6.1, reproduces Depauw's regularity result from [Dep98] using methods more closely mirroring Serfati's approach, as developed by Bae and

Kelliher in [BK15, BK21]. Serfati's approach adapts nicely to the bounded domain problem since many of the arguments can be adapted by relatively simple calculations bounding the extra terms arising from the corrector term in the Biot-Savart kernel for a bounded domain. Though it is not the first proof that solves the bounded domain vortex patch problem specifically, the approach is useful because it yields a more flexible result that can be applied to a wider variety of problems, which will be discussed in Chapter 4.

1.6 The Main Result

The result is achieved using Chemin's idea of a sufficient family of vector fields. This is a family $\mathcal{Y} = \left(Y^{(\lambda)}\right)_{\lambda \in \Lambda}$ of vector fields indexed by Λ that, roughly speaking, are used to measure the directions in which the vorticity is well-behaved in the sense that the vorticity is Hölder continuous in directions tangent to $\mathcal{Y}$. The required properties of this family are that they never simultaneously vanish at any point and that the vector fields and their divergences are sufficiently smooth. More precisely, for an open set Ω in $\mathbb{R}^2$, we define

$$I_\Omega(\mathcal{Y}) := \inf_{x \in \Omega} \sup_{\lambda \in \Lambda} \left| Y^{(\lambda)} \right|.$$

Following the example of [BK15], we call $\mathcal{Y}$ a *sufficient* $C^\alpha(\Omega)$ *family* of vector fields on an open set Ω when

$$\mathcal{Y} \in C^\alpha(\Omega), \quad \operatorname{div} \mathcal{Y} \in C^\alpha(\Omega), \quad \text{and } I_\Omega(\mathcal{Y}) > 0. \tag{1.9}$$

The notation used in the first two conditions is defined in Section 2.1 to mean that $Y^{(\lambda)}$ and $\operatorname{div} Y^{(\lambda)}$ are in $C^\alpha(\Omega)$ for all $\lambda \in \Lambda$. We note that this definition is a modification of that used by Chemin in [Che91, Che93].

We would like to have a meaningful way for a sufficient family to evolve from time 0 to an arbitrary time t and remain a sufficient family. To do so, we use the flow maps $\eta(t,x)$ defined in (1.7). For any fixed time t, the flow maps $\eta_t := \eta(t,\cdot) : \Omega \to \Omega$ are diffeomorphisms, so they give rise to unique pushforwards of vector fields by η_t ([Lee13, Chapter 3]). Let Y_0 be a vector field on $\mathbb{R}^2$ at time $t = 0$. We define the pushforward of Y_0 to time t to be

$$Y\left(t, \eta(t,x)\right) := \left(Y_0(x) \cdot \nabla\right) \eta(t,x). \tag{1.10}$$

This is simply the Jacobian of the diffeomorphism $\eta(t,\cdot)$ multiplied by Y_0. By relabeling $x = \eta^{-1}(t, \widetilde{x})$, we have the following equivalent definition that will be useful:

$$Y(t,x) := \left(Y_0\left(\eta^{-1}(t,x)\right) \cdot \nabla\right) \eta\left(t, \eta^{-1}(t,x)\right). \tag{1.11}$$

Part of Chemin's strategy was to show that, for all time, the pushforward $\mathcal{Y}(t)$ of a sufficient family $\mathcal{Y}_0$ at time $t = 0$ remains a sufficient family, where $\mathcal{Y}(t)$ is defined by (2.6) as the family of pushforwards of members of $\mathcal{Y}_0$.

With these definitions, we can now state the main result:

Theorem 1.6.1 (Main Result) *Let Ω be a bounded simply connected domain in $\mathbb{R}^2$ with a C^∞ boundary. Let $\mathcal{Y}_0$ be a sufficient $C^\alpha(\Omega)$ family of vector fields on Ω. Suppose that $\omega_0 = \operatorname{curl} u_0 \in L^\infty(\Omega)$, $\operatorname{dist}(\operatorname{supp} \omega_0, \partial\Omega) > 0$, and that $\mathcal{Y}_0 \cdot \nabla u_0 \in C^\alpha(\Omega)$. Then for all time $T > 0$, there exists a unique solution to the Euler equations (1.1) through (1.3) with*

$\mathcal{Y} \cdot \nabla u \in L^\infty(0,T;C^\alpha(\Omega))$. *Furthermore, the following estimates hold for all time* $t \in [0,T]$*:*

$$\|\nabla u(t,\cdot)\|_{L^\infty(\Omega)} \le Ce^{Ct}, \tag{1.12}$$

$$\|\mathcal{Y}(t,\cdot)\|_{C^\alpha(\Omega)} \le Ce^{Ce^{Ct}}, \tag{1.13}$$

$$\|\operatorname{div}\mathcal{Y}(t,\cdot)\|_{C^\alpha(\Omega)} \le Ce^{Ce^{Ct}}, \tag{1.14}$$

$$\|\operatorname{div}(\omega\mathcal{Y})(t,\cdot)\|_{C^{\alpha-1}(\Omega)} \le Ce^{Ce^{Ct}}, \tag{1.15}$$

$$\|\mathcal{Y} \cdot \nabla u(t,\cdot)\|_{C^\alpha(\Omega)} \le Ce^{Ce^{Ct}}, \tag{1.16}$$

$$\|\nabla \eta(t,\cdot)\|_{L^\infty(\Omega)}, \left\|\nabla \eta^{-1}(t,\cdot)\right\|_{L^\infty(\Omega)} \le Ce^{Ce^{Ct}}, \tag{1.17}$$

$$I_\Omega(\mathcal{Y})(t) \ge I_\Omega(\mathcal{Y}_0)\, e^{-Ce^{Ct}}, \tag{1.18}$$

where the constant C *depends only on* Ω*,* α*,* u_0*,* $\mathcal{Y}_0$*, and* T*.*

Here, $L^\infty(0,T;C^\alpha(\Omega))$ is the space of functions $f : [0,T] \to C^\alpha(\Omega)$ such that the supremum over $t \in [0,T]$ of $\|f(t)\|_{C^\alpha(\Omega)}$ is finite. We note that the inclusion $\mathcal{Y} \cdot \nabla u \in L^\infty(0,T;C^\alpha(\Omega))$ is implied by the estimate (1.16).

For the majority of the proof, we assume that Ω is the open unit disk $B(0,1)$. The fact that the theorem can then be extended to any bounded simply connected domain with a C^∞ boundary follows primarily because the properties of the Biot-Savart kernel are not changed significantly between the disk and such a domain. The details of this will be discussed in Section 3.6.

The hypothesis that the initial vorticity ω_0 is compactly supported in Ω also appears in Depauw's result from [Dep98]. We make this assumption because the Biot-Savart kernel K_Ω for a bounded domain is singular along the boundary. This makes it difficult to obtain an initial estimate on the quantity $\|\nabla u(t,\cdot)\|_{L^\infty}$, which we derive from the Biot-Savart law that recovers the velocity field from the vorticity through $u(t,x) = \int_\Omega K_\Omega(x,y)\omega(t,y)dy$. The Biot-Savart Laws in both the full plane and in the unit disk are discussed in Section 2.5. The initial estimate of $\|\nabla u(t,\cdot)\|_{L^\infty}$ is given by (2.21) in Section 2.6. If the vorticity is non-zero on the boundary of the domain, the bound (2.21) could diverge. However, if the initial vorticity ω_0 is supported away from the boundary, then properties of the flow map guarantee that the vorticity $\omega(t,x)$ at any time t will also be supported away from the boundary, allowing an estimate to be obtained.

The rest of this work will now proceed as follows. In Chapter 2, we set out the notation, conventions, and definitions used in this work, review much of the necessary background material, and obtain an initial estimate on the gradient of the velocity u through the Biot-Savart law. In Chapter 3, we present the proof of Theorem 1.6.1. In Chapter 4, we present some sets of initial data satisfying the hypotheses of Theorem 1.6.1, including showing how Theorem 1.6.1 proves that classical vortex patches in a simply connected bounded domain maintain their boundary regularity assuming that the initial vorticity is zero in a neighborhood of the domain's boundary. We then discuss some possible avenues of future work to expand the results presented here to more general situations. The appendix lists some fundamental results from the theory of ordinary differential equations that are used throughout this work along with the proof of a lemma from Chapter 3.

Chapter 2

Preliminaries

2.1 Notation, Conventions, and Definitions

We now fix the notation that will be used throughout. A point $x \in \mathbb{R}^2$ is represented by the ordered pair (x_1, x_2). For a vector u, we write u^i to denote the ith component. For a matrix M, we write M^i_j to denote the entry in the ith row and jth column and denote the transpose of M by M^T. Subscripts will be used to denote partial derivatives with respect to spatial variables. For example, $\partial_2 f := \partial_{x_2} f$, and $\partial_1 u^2$ represents the derivative of the second component of the vector u with respect to the first spatial variable x_1.

We define ∇u, the Jacobian matrix of u, to be the 2×2 matrix with entries given by

$$(\nabla u)^i_j = \partial_j u^i.$$

For a point $x = (x_1, x_2)$, we write $x^\perp$ to mean $(-x_2, x_1)$. We will also use the perpendicular gradient operator, defined by $\nabla^\perp = (-\partial_2, \partial_1)$. For $u = (u^1, u^2)$, we will use the scalar curl defined by $\operatorname{curl} u := \partial_1 u^2 - \partial_2 u^1$. This is simply the third component of the standard vector curl if we view u as three-dimensional by $u = (u^1, u^2, 0)$. We follow the common convention that gradient and divergence operators apply only to the spatial variables (and not the time variable). We will use x subscripts when we want to explicitly note that the operations are with respect to the x variable only, such as ∇_x and div_x.

We will use $\mathbb{1}_U$ to denote the indicator function of U, that is, the function that is identically 1 on U and zero elsewhere. We will use $B(x,r)$ to denote the open disk centered at $x \in \mathbb{R}^2$ of radius $r > 0$. For $U \subseteq \mathbb{R}^2$, a measurable integral kernel $L : U \times U \to \mathbb{R}$, and a measurable function $f : U \to \mathbb{R}$, we define the integral transform

$$L[f](x) := \text{p. v.} \int_U L(x,y) f(y)\, dy := \lim_{r \to 0^+} \int_{U \setminus B(x,r)} L(x,y) f(y)\, dy, \tag{2.1}$$

provided the limit exists.

We will write $C(p_1, \ldots, p_n)$ to denote that a constant depends only on the parameters $p_1, \ldots, p_n$. We follow the convention that such constants can vary from expression to expression and even between two occurrences within the same expression.

Throughout this document, we fix the Hölder exponent $\alpha \in (0,1)$. We will write $|v|$ for the Euclidean norm of $v = \left(v^1, v^2\right)$ defined by $|v|^2 = \left(v^1\right)^2 + \left(v^2\right)^2$. For a 2×2 matrix M, we will use the operator norm

$$|M| := \max_{|v|=1} |Mv| \,.$$

If X is a function space, we define

$$\|v\|_X := \| |v| \|_X \,, \qquad \|M\|_X := \| |M| \|_X \,.$$

We will now define the various function spaces that we will be using. First, for any function $f \in L^1 \cap L^\infty$, we define the norm

$$\|f\|_{L^1 \cap L^\infty} := \|f\|_{L^1} + \|f\|_{L^\infty} . \tag{2.2}$$

Note that, by Lebesgue space interpolation, for any $p \in (1, \infty)$, we have $L^1 \cap L^\infty \subseteq L^p$ with $\|f\|_{L^p} \leq \|f\|_{L^1}^{\frac{1}{p}} \|f\|_{L^\infty}^{1-\frac{1}{p}}$.

Definition 2.1.1 (Hölder spaces) *Let $\alpha \in (0,1)$ and $U \subseteq \mathbb{R}^2$ be open. We have the following standard ([Eva10, Section 5.1]) Hölder α-seminorm and α-norm, respectively, for a function f defined on U*[1]*:*

$$\|f\|_{\dot{C}^\alpha(U)} = \sup_{x,y \in U,\, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha},$$

$$\|f\|_{C^\alpha(U)} = \|f\|_{L^\infty(U)} + \|f\|_{\dot{C}^\alpha(U)} .$$

The Hölder space $C^\alpha(U)$ is the space of all functions f with $\|f\|_{C^\alpha(U)} < \infty$.

For $k \in \mathbb{Z}^+$ and a k-times continuously differentiable function f, we define

$$\|f\|_{C^{k,\alpha}(U)} = \sum_{|\beta| \leq k} \left\| D^\beta f \right\|_{L^\infty(U)} + \sum_{|\beta| = k} \left\| D^\beta f \right\|_{C^\alpha(U)}$$

and denote by $C^{k,\alpha}(U)$ the space of all such functions with $\|f\|_{C^{k,\alpha}(U)} < \infty$. The space $C^{k,\alpha}(U)$ is also sometimes written as $C^{k+\alpha}(U)$.

We define the negative Hölder space $C^{\alpha-1}(U)$ by

$$C^{\alpha-1}(U) = \{f + \operatorname{div} v : f, v \in C^\alpha(U)\} \tag{2.3}$$

and use the norm

$$\|h\|_{C^{\alpha-1}(U)} = \inf\{\|f\|_{C^\alpha(U)} + \|v\|_{C^\alpha(U)} : h = f + \operatorname{div} v; f, v \in C^\alpha(U)\}, \tag{2.4}$$

[1] Note that the functions in $C^\alpha(U)$ and $C^{k,\alpha}(U)$ are uniformly continuous and so can be uniquely extended to the closure $\overline{U}$.

noting that the divergence above is interpreted in the distributional sense since a function $v \in C^\alpha$ is not necessarily differentiable.

Note that, by letting $f = 0$ and $h = \operatorname{div} v$ in the above definition of $C^{\alpha-1}$, we immediately get the inequality

$$\|\operatorname{div} v\|_{C^{\alpha-1}(U)} \leq \|v\|_{C^\alpha(U)}. \tag{2.5}$$

Another useful immediate observation is that, for any weakly differentiable function $u \in C^\alpha(U)$, each of its (distributional) partial derivatives satisfy $\partial_i u \in C^{\alpha-1}$.

Definition 2.1.2 (Sobolev spaces) *Let k be a non-negative integer, $1 \leq p \leq \infty$, and $U \subseteq \mathbb{R}^2$ be open. The Sobolev space $W^{k,p}(U)$ is defined as the set of all locally integrable functions f such that, for each multi-index α with $|\alpha| \leq k$, the weak derivative $D^\alpha f$ exists and is a member of $L^p(U)$. We have the standard ([Eva10, Section 5.2]) Sobolev space norms*

$$\|f\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \int_U |D^\alpha f|^p \, dx\right)^{1/p} & \text{for } 1 \leq p < \infty, \\ \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^\infty(U)} & \text{for } p = \infty. \end{cases}$$

For $p = 2$, we also use the notation $H^k(U) := W^{k,2}(U)$.

We defined sufficient families of vector fields in (1.9). We will repeat the definition here for completeness and fix some notation that will be used when working with such families. Let $\mathcal{Y} = \left(Y^{(\lambda)}\right)_{\lambda\in\Lambda}$ be a family of vector fields on Ω indexed by Λ. For any function f on vector fields (such as div), we define $f(\mathcal{Y})$ to be the family of images under f of the members of $\mathcal{Y}$:

$$f(\mathcal{Y}) := \left(f\left(Y^{(\lambda)}\right)\right)_{\lambda\in\Lambda}.$$

For any Banach space X, we define

$$\|f(\mathcal{Y})\|_X := \sup_{\lambda\in\Lambda} \left\| f\left(Y^{(\lambda)}\right)\right\|_X .$$

When $\|f(\mathcal{Y})\|_X < \infty$, we say that $f(\mathcal{Y}) \in X$. We also define

$$I(\mathcal{Y}) := \inf_{x\in\mathbb{R}^2} \sup_{\lambda\in\Lambda} \left|Y^{(\lambda)}\right| .$$

We define the pushforward of $\mathcal{Y}$ to be

$$\mathcal{Y}(t,\cdot) = \left(Y^{(\lambda)}(t,\cdot)\right)_{\lambda\in\Lambda}, \tag{2.6}$$

where $Y^{(\lambda)}(t,\eta(t,x)) := \left(Y_0^{(\lambda)}(x)\cdot\nabla\right)\eta(t,x)$ as in (1.10).

We will use the following standard mollifiers:

Definition 2.1.3 (Mollifiers) *Let $\rho \in C_c^\infty(\mathbb{R}^2)$ with $\rho \geq 0$ have $\|\rho\|_{L^1} = 1$ and be radially symmetric. For example, a multiple of $\rho(x) = e^{-\frac{1}{1-|x|^2}}$, extended by zero outside the unit disk, is suitable. For $n \in \mathbb{N}$, define*

$$\rho_n(x) = n^2\rho(nx).$$

Note that, for all $n \in \mathbb{N}$, ρ_n is supported in $B(0,1/n)$ and $\|\rho_n\|_{L^1} = 1$.

We take a radially symmetric function $a \in C_c^\infty(\mathbb{R}^2)$ taking values in $[0,1]$ with $a = 1$ on $B(0,1)$ and $a = 0$ on $B(0,2)^c$. For $r > 0$, we define the rescaled cutoff function

$$a_r(x) = a(x/r). \tag{2.7}$$

Note that this gives the useful properties that $a_r = 1$ on $B(0,r)$ and $a_r = 0$ outside of $B(0,2r)$.

We will be using two extension operators taken from [Ste70]. The first one extends functions from Sobolev spaces on Ω to the corresponding Sobolev space on $\mathbb{R}^2$, and the second one extends functions from Hölder spaces on Ω to the corresponding Hölder space on $\mathbb{R}^2$. Their constructions can be found in Chapter VI of [Ste70] as Theorem 5 and Theorem 3, respectively. See also [Eva10, Section 5.4].

Lemma 2.1.4 (Stein Sobolev space extension) *Let Ω be a bounded open set in $\mathbb{R}^n$ with C^1 boundary. There exists a bounded linear extension operator $\mathcal{E}$, which we call the Stein extension operator, so that for any $1 \leq p \leq \infty$ and any non-negative integer k, $\mathcal{E} : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^2)$ and has the following properties:*

1. $(\mathcal{E}f)\big|_{\Omega} = f$,

2. $\|\mathcal{E}f\|_{W^{k,p}(\mathbb{R}^2)} \leq C \|f\|_{W^{k,p}(\Omega)}$ *with the constant C depending only on p, k, and Ω.*

We note the remarkable property that the above Stein extension operator itself is independent of k and p, depending only on Ω, and simultaneously extends all functions in any Sobolev space on Ω to $\mathbb{R}^2$.

Lemma 2.1.5 (Stein Hölder space extension) *Let Ω be a bounded open set in $\mathbb{R}^n$ with C^1 boundary. There exists a bounded linear extension operator $\mathcal{E}_H$, which we also call the Stein extension operator, so that for any $0 < \alpha \leq 1$, $\mathcal{E}_H : C^{\alpha}(\Omega) \longrightarrow C^{\alpha}(\mathbb{R}^2)$ and has the following properties:*

1. $(\mathcal{E}_H f)\big|_{\Omega} = f$,

2. $\|\mathcal{E}_H f\|_{C^{\alpha}(\mathbb{R}^2)} \leq C \|f\|_{C^{\alpha}(\Omega)}$ *with the constant C depending only on α and Ω.*

2.2 Useful Inequalities

We begin with two well-known inequalities on Sobolev spaces. They can be found, for instance, as Theorem 3 in [Eva10, Section 5.6] and Theorem 1 in [Eva10, Section 5.8], respectively.

Lemma 2.2.1 (Poincaré Inequality) *Let Ω be a bounded open subset of $\mathbb{R}^n$. Suppose that $1 \leq p \leq \infty$ and that $f \in W^{1,p}(\Omega)$ is compactly supported. Then there exists a constant C, depending only on n, p, and Ω, such that*

$$\|f\|_{L^p(\Omega)} \leq C \|\nabla f\|_{L^p(\Omega)}.$$

Lemma 2.2.2 (Poincaré-Wirtinger Inequality) *Let Ω be a bounded, connected, open subset of $\mathbb{R}^n$ with C^1 boundary. Denote the average value of a function f on Ω by $\bar{f} = |\Omega|^{-1} \int_\Omega f(y)\,dy$. Suppose that $1 \leq p \leq \infty$. Then there exists a constant C, depending only on n, p, and Ω, such that*

$$\|f - \bar{f}\|_{L^p(\Omega)} \leq C \|\nabla f\|_{L^p(\Omega)}$$

for each function $f \in W^{1,p}(\Omega)$. Note that, if $\bar{f} = 0$, then we have

$$\|f\|_{L^p(\Omega)} \leq C \|\nabla f\|_{L^p(\Omega)}.$$

In light of the conclusion of Lemma 2.2.2, the following result is useful. We will apply this to the fluid velocity field in Section 2.7.

Lemma 2.2.3 *Let Ω be as in Lemma 2.2.2 and let $f \in H^1(\Omega)$ be a vector field with $\operatorname{div} f = 0$ and $f \cdot \hat{n} = 0$ on $\partial\Omega$. Then $\int_\Omega f^i\,dx = 0$ for $i = 1, 2$. In particular, $\bar{f} = 0$.*

Proof. Using integration by parts, we can directly calculate that

$$\int_{\Omega} f^i \, dx = \int_{\Omega} f \cdot \nabla x^i \, dx = -\int_{\Omega} (\operatorname{div} f) x^i \, dx + \int_{\partial\Omega} (f \cdot \hat{n}) x^i \, dx = 0,$$

where the last two integrals are zero by hypothesis. We note that the boundary integral is well-defined since, for $f \in H^1(U)$, the trace Tf of f along ∂U is in $H^{1/2}(U)$ ([Leo09, Section 15.3]). Since the average value of a vector field can be computed component-wise, the average value of f is zero in Ω. ■

We next give a useful estimate of the Lipschitz constant of a function in terms of its gradient.

Lemma 2.2.4 *If f is differentiable almost everywhere on a convex domain $\Omega \subseteq \mathbb{R}^n$, then for all $x \neq y \in \Omega$,*

$$\frac{|f(x) - f(y)|}{|x - y|} \leq \|\nabla f\|_{L^\infty(\Omega)} .$$

Proof. For any $x \neq y$, the Fundamental Theorem of Calculus and the Chain Rule give that

$$\begin{aligned} f(x) - f(y) &= \int_0^1 \frac{d}{ds} f(sx + (1-s)y) \, ds \\ &= \int_0^1 \nabla f(sx + (1-s)y) \cdot (x - y) \, ds. \end{aligned}$$

Thus, we have that

$$|f(x) - f(y)| \leq \|\nabla f\|_{L^\infty(\Omega)} |x - y| \int_0^1 ds$$

so that the claim now follows. ■

Note that Lemma 2.2.4 requires the domain to be convex so that the expression $f(sx + (1-s)y)$ used in the proof is well-defined. However, if the domain is bounded, we can obtain the following analogous result.

Lemma 2.2.5 *If $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with C^1 boundary and $f \in W^{1,\infty}(\Omega)$, then for all $x \neq y \in \Omega$,*

$$\frac{|f(x) - f(y)|}{|x - y|} \leq C(\Omega) \left\| \nabla f \right\|_{L^\infty(\Omega)}.$$

Proof. Since $f \in W^{1,\infty}(\Omega)$, we can employ the Stein extension operator from Lemma 2.1.4 to extend f to the function $\mathcal{E}f$ defined on $\mathbb{R}^2$. Since $\mathbb{R}^2$ is convex, we can use Lemma 2.2.4, the definition of the Sobolev space norm, and the properties of the Stein extension to see that

$$\begin{aligned}
\frac{|f(x) - f(y)|}{|x - y|} &= \frac{|\mathcal{E}f(x) - \mathcal{E}f(y)|}{|x - y|} \\
&\leq \left\| \nabla(\mathcal{E}f) \right\|_{L^\infty(\mathbb{R}^2)} \\
&\leq \left\| \mathcal{E}f \right\|_{W^{1,\infty}(\mathbb{R}^2)} \\
&\leq C(\Omega) \left\| f \right\|_{W^{1,\infty}(\Omega)} \\
&= C(\Omega) \left(\left\| f \right\|_{L^\infty(\Omega)} + \left\| \nabla f \right\|_{L^\infty(\Omega)} \right).
\end{aligned}$$

Because $|f(x) - f(y)| = \left|(f - \bar{f})(x) - (f - \bar{f})(y)\right|$, we can apply the above calculation to the function $f - \bar{f}$ to find that

$$\begin{aligned}
\frac{|f(x) - f(y)|}{|x - y|} &= \frac{\left|(f - \bar{f})(x) - (f - \bar{f})(y)\right|}{|x - y|} \\
&\leq C(\Omega) \left(\left\| f - \bar{f} \right\|_{L^\infty(\Omega)} + \left\| \nabla \left(f - \bar{f}\right) \right\|_{L^\infty(\Omega)} \right) \\
&= C(\Omega) \left(\left\| f - \bar{f} \right\|_{L^\infty(\Omega)} + \left\| \nabla f \right\|_{L^\infty(\Omega)} \right),
\end{aligned}$$

since $\bar{f}$ is constant. By Lemma 2.2.2, $\left\| f - \bar{f} \right\|_{L^\infty(\Omega)} \leq C \left\| \nabla f \right\|_{L^\infty(\Omega)}$, which completes the proof. ∎

We will need the following inequality on Hölder spaces.

Lemma 2.2.6 *Let $f \in C^{\alpha}(\Omega)$ and $g \in C^1(\Omega)$, where Ω is a bounded domain with C^1 boundary. Then we have*

$$\|f \circ g\|_{\dot{C}^{\alpha}(\Omega)} \leq C(\alpha, \Omega) \|f\|_{\dot{C}^{\alpha}(\Omega)} \|\nabla g\|^{\alpha}_{L^{\infty}(\Omega)} .$$

Proof. For $x \neq y \in \Omega$, we have

$$\begin{aligned} |(f \circ g)(x) - (f \circ g)(y)| &= \left(\frac{|f(g(x)) - f(g(y))|}{|g(x) - g(y)|^{\alpha}} \right) \left(\frac{|g(x) - g(y)|^{\alpha}}{|x - y|^{\alpha}} \right) |x - y|^{\alpha} \\ &\leq \left(\sup_{\substack{X, Y \in \Omega \\ X \neq Y}} \frac{|f(X) - f(Y)|}{|X - Y|^{\alpha}} \right) \left[C(\Omega) \|\nabla g\|_{L^{\infty}(\Omega)} \right]^{\alpha} |x - y|^{\alpha} , \end{aligned}$$

where we used Lemma 2.2.5 to bound the middle factor. This means that, for all $x, y \in \Omega$,

$$\frac{|(f \circ g)(x) - (f \circ g)(y)|}{|x - y|^{\alpha}} \leq C(\alpha, \Omega) \|f\|_{\dot{C}^{\alpha}(\Omega)} \|\nabla g\|^{\alpha}_{L^{\infty}(\Omega)} ,$$

which gives the desired inequality. ∎

We note that if Ω is convex, we could apply Lemma 2.2.4 instead of Lemma 2.2.5 in the proof to obtain $C = 1$.

The following two Grönwall Inequalities will be needed. While the standard Grönwall Inequality is a well-known classical result, we will be using a general form of the lemma (more general than the version that appears in [Eva10], for example) and the Reverse Grönwall Inequality is less well-known, so their proofs are presented here.

Lemma 2.2.7 (Grönwall's Inequality) *Suppose $h \geq 0$ is a continuous nondecreasing function on $[0, T]$, $f \geq 0$ is continuous, and $g \geq 0$ is integrable on $[0, T]$. If*

$$f(t) \leq h(t) + \int_0^t g(s) f(s)\, ds$$

for all $t \in [0,T]$*, then*

$$f(t) \le h(t)e^{\int_0^t g(s)\,ds}$$

for all $t \in [0,T]$*.*

Proof. Let $J(t) := \int_0^t g(s)f(s)\,ds$. Observe that J satisfies the differential equation $J'(t) = g(t)f(t)$. Since $g \ge 0$ and $gf \ge 0$, by hypothesis we have

$$J'(t) \le g(t)\left[h(t) + \int_0^t g(s)f(s)\,ds\right]$$

which can be rearranged as

$$J'(t) - g(t)J(t) \le g(t)h(t).$$

After multiplying both sides by the (positive) integrating factor $e^{-\int_0^t g(\tau)\,d\tau}$ and recognizing the left side as the result of the product rule, we see that, for all $t \in [0,T]$,

$$\frac{d}{dt}\left[J(t)e^{-\int_0^t g(\tau)\,d\tau}\right] \le g(t)h(t)e^{-\int_0^t g(\tau)\,d\tau}.$$

By integrating both sides of this inequality from 0 to t and noting that $J(0) = 0$, after some rearranging we see that

$$\begin{aligned} J(t) &\le e^{\int_0^t g(\tau)\,d\tau}\int_0^t g(s)h(s)e^{-\int_0^s g(\tau)\,d\tau}\,ds \\ &= \int_0^t g(s)h(s)e^{\int_s^0 g(\tau)\,d\tau}e^{\int_0^t g(\tau)\,d\tau}\,ds \\ &= \int_0^t g(s)h(s)e^{\int_s^t g(\tau)\,d\tau}\,ds. \end{aligned}$$

So, by hypothesis, we have

$$f(t) \le h(t) + J(t) \le h(t) + \int_0^t g(s)h(s)e^{\int_s^t g(\tau)\,d\tau}\,ds.$$

Since h is non-decreasing, we can replace $h(s)$ in this last inequality to obtain

$$\begin{aligned} f(t) &\leq h(t) + h(t)\int_0^t g(s)e^{\int_s^t g(\tau)\,d\tau}\,ds \\ &= h(t) + h(t)\left[-\int_0^t \left(\frac{d}{ds}e^{-\int_t^s g(\tau)\,d\tau}\right)\,ds\right] \\ &= h(t) - h(t)\left[e^{\int_t^t g(\tau)\,d\tau} - e^{\int_0^t g(\tau)\,d\tau}\right] \\ &= h(t) - h(t)\left[1 - e^{\int_0^t g(\tau)\,d\tau}\right] \\ &= h(t)e^{\int_0^t g(s)\,ds}, \end{aligned}$$

as desired, where we used the Chain Rule and the Fundamental Theorem of Calculus in the second and third lines, respectively. ∎

Lemma 2.2.8 (Reverse Grönwall's Inequality) *Suppose $f > 0$ is a differentiable function on $[0,T]$ and $g \geq 0$ is integrable on $[0,T]$. If*

$$f'(t) \geq -g(t)f(t)$$

for all $t \in [0,T]$, then

$$f(t) \geq f(0)e^{-\int_0^t g(s)\,ds}$$

for all $t \in [0,T]$.

Proof. Note that, by hypothesis,

$$\frac{d}{dt}\ln f(t) = \frac{f'(t)}{f(t)} \geq -g(t).$$

By integrating the inequality from 0 to t, for any $t \in [0,T]$, we see that

$$\ln f(t) - \ln f(0) = \ln\frac{f(t)}{f(0)} \geq -\int_0^t g(s)\,ds,$$

which can be exponentiated to yield

$$\frac{f(t)}{f(0)} \geq e^{-\int_0^t g(s)\,ds},$$

from which the desired inequality follows. ■

2.3 The Two-Dimensional Euler Equations

We now review some of the basic properties related to the Euler equations in two dimensions, which were given above in (1.1) through (1.3):

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{2.8}$$

These equations model the flow of an ideal (incompressible and inviscid) homogeneous fluid in $\mathbb{R}^2$ and can be derived from the conservation of momentum for a continuum, as in [Mey82, Section 12]. The unknowns here are $u(t,x)$ and $p(t,x)$ which represent the velocity and pressure, respectively, of the fluid at time $t \geq 0$ and position $x \in \mathbb{R}^2$. The second equation describes the incompressibility of the fluid and the third equation gives the divergence-free initial velocity. In a bounded domain U, we will also consider the boundary condition that

$$u \cdot \hat{n} = 0, \tag{2.9}$$

where $\hat{n}$ is the outward unit normal to the boundary ∂U. This ensures that no fluid flows in or out of the domain through the boundary.

The vorticity is defined as $\omega(u) := \operatorname{curl} u$. Taking the scalar curl of equations (2.8) yields the vorticity-stream formulation of the two-dimensional incompressible Euler equations ([MB02, Section 2.1]):

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ \omega(0,x) = \omega_0(x). \end{cases} \tag{2.10}$$

As briefly discussed in Chapter 1 (see (1.7)), associated to the fluid velocity are particle trajectories $\eta(t,x)$, also called flow maps, defined by

$$\begin{cases} \partial_t \eta(t,x) = u(t, \eta(t,x)), \\ \eta(0,x) = x. \end{cases} \tag{2.11}$$

According to classical ODE theory (see the Appendix, Theorem A.1), the system (2.11) will have a unique solution η for all time if the velocity u is Lipschitz continuous in space, uniformly in time. When the initial vorticity $\omega_0 \in L^1 \cap L^\infty$, the associated velocity field is not necessarily Lipschitz, but is instead log-Lipschitz ([MB02, Lemma 8.1]) so admits an Osgood modulus of continuity and so (2.11) has a unique solution (see Theorems A.3 and A.4 and Remark A.5). Specifically, initial vorticities satisfying the assumptions of Theorem 1.6.1 give rise to unique particle trajectories. The flow maps are continuous in time and are diffeomorphisms for any fixed time t. The divergence-free property of u guarantees that the diffeomorphisms are measure-preserving. (For instance, Proposition 1.4 of [MB02] states that the fluid incompressibility, the divergence-free condition, and the Jacobian determinant of η equaling one are all equivalent.) The regularity of the flow map in space is the same as that of the velocity field.

2.4 Weak Solutions and Well-Posedness

The equations (2.10) completely describe the fluid behavior and solving for the vorticity $\omega(t,x)$ uniquely determines the velocity field through the Biot-Savart Law, discussed in detail below in Section 2.5. However, the equations require the vorticity to possess more regularity than simply being in L^∞, which is often too restrictive an assumption to make. For instance, even a simple vortex patch such as $\omega_0(x) = \mathbb{1}_{B(0,1)}$ cannot be studied in this framework. What is needed is an equivalent expression of the vorticity-stream formulation that allows vorticities that are in the less restrictive natural class $L^1 \cap L^\infty$. This motivates the definition of a weak solution, which we present here from [MB02, Section 8.2].

Definition 2.4.1 (Weak Solutions to the 2D Euler Equations) *Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected bounded domain. Given $\omega_0 \in L^\infty(\Omega)$, the velocity-vorticity pair (u,ω) is a weak solution to the vorticity-stream formulation of the two-dimensional Euler equations in Ω with initial data $\omega_0(x)$ provided that*

(i) $\omega \in L^\infty(0,T;L^\infty(\Omega))$,

(ii) *u can be recovered from ω via the Biot-Savart Law (Theorem 2.5.2),*

(iii) *for all* $\varphi \in C^1\left(0,T;C^1_c(\Omega)\right)$,

$$\int_\Omega \varphi(T,x)\omega(T,x)\,dx - \int_\Omega \varphi(0,x)\omega_0(x)\,dx = \int_0^T \int_\Omega \frac{D\varphi}{Dt}\omega\,dx\,dt. \tag{2.12}$$

While the initial vorticity for a classical vortex patch is discontinuous at the boundary of the patch, it is nonetheless integrable and bounded, making weak solutions the natural choice to which we restrict our attention.

For smooth initial data, the existence of classical smooth solutions to the two-dimensional Euler equations for all time was proved by Wolibner in [Wol33]. The existence and uniqueness of weak solutions in the plane for bounded initial vorticity was proved by Yudovich in [Yud63].

2.5 The Biot-Savart Law

The Biot-Savart Law is fundamental in the study of the two-dimensional Euler equations. In this section, we will first derive the Biot-Savart Law for the whole plane $\mathbb{R}^2$ and then use it to derive the Biot-Savart Law for the unit disk $\Omega = B(0,1)$.

Our goal is to be able to recover the velocity $u(t,x)$ from the vorticity $\omega(t,x)$. What follows is valid for any fixed time t, so we will suppress the time argument for simplicity. We start by considering the following Poisson equation in the unknown $\varphi(x) := \varphi(t,x)$:

$$\begin{cases} \Delta\varphi = \omega & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.13}$$

If $\varphi \in H^1(\Omega)$ satisfies (2.13), define the vector field $v := \nabla^\perp \varphi$. Since $\omega \in L^2(\Omega)$, elliptic regularity theory ([Eva10, Section 6.3, Theorem 2]) gives that $\varphi \in H^2(\Omega)$. Then we would have

$$\operatorname{div} v = \partial_1(-\partial_2\varphi) + \partial_2(\partial_1\varphi) = 0$$

and

$$\operatorname{curl} v = \partial_1(\partial_1\varphi) - \partial_2(-\partial_2\varphi) = \Delta\varphi = \omega.$$

Because $\varphi = 0$ on the boundary of Ω, $\nabla^{\perp}\varphi = v$ is tangential to the boundary, so that $v \cdot \hat{n} = 0$ on $\partial\Omega$. Thus, v is a vector field in Ω that has the same divergence, curl, and normal boundary condition as the velocity field u. We now show that $v = u$.

Let $w := u - v$. Then w has zero divergence, zero curl, and $w \cdot \hat{n} = 0$ on $\partial\Omega$. Because w is irrotational and Ω is simply connected, there exists a scalar potential f for w so that $w = \nabla f$. Since w is divergence-free, $\operatorname{div}(w) = \operatorname{div}(\nabla f) = \Delta f = 0$. Because $w \cdot \hat{n} = 0$ on $\partial\Omega$, we have $\nabla f \cdot \hat{n} = \frac{\partial f}{\partial \hat{n}} = 0$. Recall the vector identity $\operatorname{div}(f\nabla f) = \nabla f \cdot \nabla f + f\Delta f$. Then we can calculate that

$$\begin{aligned}
\int_{\Omega} |\nabla f|^2 \, dx &= \int_{\Omega} \nabla f \cdot \nabla f \, dx \\
&= \int_{\Omega} \operatorname{div}(f\nabla f) \, dx - \int_{\Omega} f\Delta f \, dx \\
&= \int_{\partial\Omega} f \frac{\partial f}{\partial \hat{n}} - \int_{\Omega} f\Delta f \, dx \\
&= 0,
\end{aligned}$$

where we used the divergence theorem. Thus, $|\nabla f|^2 = 0$, so that $w = \nabla f = 0$ and $u = v$.

So if we solve the Poisson equation (2.13) for φ, we can write the velocity u as

$$u(t, x) = \nabla^{\perp}\varphi(t, x). \tag{2.14}$$

We call φ the *stream function* for u.

Classical potential theory ([Eva10, Section 2.1], for instance) guarantees the existence of a unique stream function $\varphi \in H^2$ decaying at infinity and satisfying $\Delta\varphi = \omega$ given by convolution of ω with the Newtonian potential in $\mathbb{R}^2$, assuming ω vanishes sufficiently rapidly at infinity:

$$\varphi(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x - y| \, \omega(t, y) \, dy, \quad x \in \mathbb{R}^2. \tag{2.15}$$

Note that $\omega \in L^1 \cap L^\infty$ is sufficient for this integral to exist. Via (2.14), we can differentiate (2.15) under the integral to obtain an equation for u in terms of ω. The resulting expression for the velocity u in terms of its vorticity ω is called the Biot-Savart Law.

In the full plane $\mathbb{R}^2$, the Biot-Savart Law states that the velocity $u(t,x)$ of a divergence-free vector field that vanishes at infinity can be recovered from the vorticity $\omega(t,x) = \operatorname{curl} u(t,x)$ through convolution with a kernel function. This lemma can be found, for instance, as Proposition 2.1 of [MB02].

Lemma 2.5.1 (Biot-Savart Law in $\mathbb{R}^2$) *Let $u(t,x)$ be the divergence-free velocity associated with the vorticity $\omega(t,x) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then for all time $t \geq 0$, we have*

$$u(t,x) = K * \omega(t,x) = \int_{\mathbb{R}^2} K(x-y)\omega(t,y)\,dy,$$

where $K(x) = \nabla^\perp G(x)$ and $G(x)$ is the fundamental solution of the Laplacian[2] *in $\mathbb{R}^2$. For the domain $\mathbb{R}^2$, we have*

$$G(x) = \frac{1}{2\pi}\ln|x|, \quad K(x) = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}. \tag{2.16}$$

Using the notation of (2.1), we can say that $K[\omega]$ is the unique divergence-free vector field vanishing at infinity whose vorticity is ω.

The function K is called the Biot-Savart kernel (for the plane). We sometimes use the notation $K(x,y)$ to mean $K(x-y)$, such as when we write $K[\omega]$. It is important to note that K is locally integrable.

[2] We are using the convention that the fundamental solution $G(x)$ satisfies $\Delta G = \delta_0$ in $\mathbb{R}^2$, where δ_0 is Dirac's delta function, and not that $G(x)$ satisfies $-\Delta G = \delta_0$ as some authors (including Evans in [Eva10]) do.

The Biot-Savart Law yields a convenient expression for the velocity gradient, obtained by applying the gradient to both sides and performing standard calculations to compute the distributional derivative of K. For instance, see either equation (1.3) of [BC93] or Proposition 2.17 of [MB02]:

$$\nabla u(x) = \frac{\omega(x)}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \text{p. v.} \int_{\mathbb{R}^2} \nabla_x K(x-y)\omega(y)\, dy. \tag{2.17}$$

We note that this identity is valid for any $\omega \in L^1 \cap L^\infty$. This expression will be used later to obtain an analogous estimate in the unit disk in the proof of Lemma 2.6.1. The principal value integral can be thought of as a singular integral operator applied to the vorticity.

We now will derive the the Biot-Savart law for the unit disk Ω, which is the basis for many of the calculations that will follow. By following an identical line of reasoning as that given above for the full plane case, we see that the only necessary difference in the Biot-Savart Law will be that we must now use Green's function for Ω to correct the fundamental solution of the Laplacian in $\mathbb{R}^2$ in order to solve the Poisson equation (2.13). So, we begin by recalling ([Str08, Chapter 7, Equation (18)]) that the fundamental solution for the Laplacian in the unit disk Ω is

$$G_\Omega(x,y) = \frac{1}{2\pi}\left(\ln|y-x| - \ln\big||x|\,|y-x^*|\big|\right),$$

where $x^* := \frac{x}{|x|^2}$ is inversion across the unit circle. As with the full plane case, the Biot-Savart kernel in Ω will be

$$K_\Omega(x,y) := \nabla_x^\perp G_\Omega(x,y).$$

Noting that $\nabla_x^\perp$ applied to the first term of G_Ω is exactly the full-plane Biot-Savart kernel K given in (2.16), we find that

$$K_\Omega(x,y) = K(x-y) - \nabla_x^\perp \left(\frac{1}{2\pi} \ln \big| |x|\, |y - x^*| \big| \right).$$

Note that, while Green's functions $G(x,y)$ are symmetric, $K(x,y)$ is not necessarily symmetric since the gradient is with respect to the x variable only. Nevertheless, we would like an alternate expression for K_Ω that inverts the y-argument instead of the x in order to make it easier to differentiate with respect to x. To obtain this, note that

$$\begin{aligned}
|x|^2 \, |y - x^*|^2 &= |x|^2 \left| y - \frac{x}{|x|^2} \right|^2 \\
&= |x|^2 \left| |y|^2 - \frac{2x \cdot y}{|x|^2} + \frac{|x|^2}{|x|^4} \right| \\
&= \left| |x|^2 \, |y|^2 - 2x \cdot y + 1 \right| \\
&= |y|^2 \left| |x|^2 - \frac{2x \cdot y}{|y|^2} + \frac{|y|^2}{|y|^4} \right| \\
&= |y|^2 \left| x - \frac{y}{|y|^2} \right|^2 \\
&= |y|^2 \, |x - y^*|^2 ,
\end{aligned}$$

so that $\ln \big| |x|\, |y - x^*| \big| = \ln \big| |y|\, |x - y^*| \big|$. Thus, we can now calculate the Biot-Savart kernel and summarize the Biot-Savart Law for the unit disk:

$$\begin{aligned}
K_\Omega(x,y) &= K(x-y) - \nabla_x^\perp \left(-\frac{1}{2\pi} \ln \big| |y|\, |x - y^*| \big| \right) \\
&= K(x-y) - \nabla_x^\perp \left(-\frac{1}{2\pi} \ln |y| \right) - \nabla_x^\perp \left(-\frac{1}{2\pi} \ln |x - y^*| \right) \\
&= K(x-y) - 0 - \left[-\frac{1}{2\pi} \left(\frac{-(x_2 - y_2^*)}{|x - y^*|^2}, \frac{(x_1 - y_1^*)}{|x - y^*|^2} \right) \right], \\
K_\Omega(x,y) &= K(x-y) - K(x - y^*). \qquad (2.18)
\end{aligned}$$

Theorem 2.5.2 (Biot-Savart Law in the Unit Disk Ω) *Let $u(t,x)$ be the divergence-free velocity associated with the vorticity $\omega(t,x) \in L^\infty(\Omega)$. Then for all time $t \geq 0$, we have*

$$u(t,x) = K_\Omega[\omega] = \int_\Omega K_\Omega(x,y)\omega(t,y)\,dy,$$

where K_Ω is defined by (2.18).

2.6 Initial Estimate of the Velocity Gradient

We will now use Theorem 2.5.2 to derive a useful expression for the spatial gradient of the fluid velocity u. We will then use this expression to obtain our initial estimate for $\|\nabla u(t,\cdot)\|_{L^\infty(\Omega)}$ that will be the basis for much of what follows. Because these calculations are valid for any fixed time t, for simplicity, we will omit the time argument for the remainder of this section.

Whenever possible, we will use established whole plane results. To this end, we will frequently find the following extension useful. Since the vorticity ω is compactly supported in Ω, it naturally extends by zero to the whole plane:

$$\widetilde{\omega}(y) := \begin{cases} \omega(y) & \text{for } y \in \Omega, \\ 0, & \text{for } y \notin \Omega. \end{cases} \tag{2.19}$$

Clearly, $\widetilde{\omega}$ has the same support and regularity as ω and, for any reasonable norm $\|\cdot\|$, $\|\widetilde{\omega}\|$ in $\mathbb{R}^2$ is the same as $\|\omega\|$ in Ω.

Lemma 2.6.1 (Expression for the Velocity Gradient) *Let u and ω be as in Theorem 2.5.2. Then for $x \in \Omega$,*

$$\nabla u(x) = \frac{\omega(x)}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \text{p. v.} \int_{\mathbb{R}^2} \nabla_x K(x-y)\widetilde{\omega}(y)\,dy - \int_\Omega \nabla_x K(x-y^*)\omega(y)\,dy$$
$$= \frac{\omega(x)}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \text{p. v.} \int_\Omega \nabla_x K_\Omega(x,y)\omega(y)\,dy.$$

Proof. By the Biot-Savart Law (Theorem 2.5.2), and assuming for now that each term in $K_\Omega(x,y)\omega(y)$ is integrable, we have that

$$\begin{aligned}\nabla u(x) &= \nabla_x \int_\Omega K_\Omega(x,y)\omega(y)\,dy \\ &= \nabla_x \int_\Omega K(x-y)\omega(y)\,dy - \nabla_x \int_\Omega K(x-y^*)\omega(y)\,dy \qquad (2.20) \\ &= \nabla_x \int_{\mathbb{R}^2} K(x-y)\widetilde{\omega}(y)\,dy - \nabla_x \int_\Omega K(x-y^*)\omega(y)\,dy.\end{aligned}$$

Because $\widetilde{\omega} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, we can apply (2.17) to the first term to obtain

$$\nabla u(x) = \frac{\omega(x)}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \text{p. v.} \int_{\mathbb{R}^2} \nabla_x K(x-y)\widetilde{\omega}(y)\,dy - \nabla_x \int_\Omega K(x-y^*)\omega(y)\,dy.$$

For the last integral, note that the singularity of $K(x-y^*)$ occurs when $x-y^* = 0$, or when $y = x^*$. Since x^* is outside of Ω for $x \in \Omega$, $K(x-y^*)\omega(y)$ is integrable (in y) on Ω and has a smooth bounded derivative with respect to x. Therefore, we can bring the gradient operator inside the integral ([Fol13, Theorem 2.27]) to obtain

$$\begin{aligned}
\nabla u(x) &= \frac{\omega(x)}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \mathrm{p.\,v.}\int_{\mathbb{R}^2} \nabla_x K(x-y)\widetilde{\omega}(y)\,dy - \int_\Omega \nabla_x K(x-y^*)\omega(y)\,dy \\
&= \frac{\omega(x)}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \mathrm{p.\,v.}\int_{\Omega} \nabla_x K(x-y)\omega(y)\,dy - \int_\Omega \nabla_x K(x-y^*)\omega(y)\,dy \\
&= \frac{\omega(x)}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \mathrm{p.\,v.}\int_{\Omega} \nabla_x K_\Omega(x,y)\omega(y)\,dy.
\end{aligned}$$

Note that the fact that each of the above integrals exists justifies our splitting of the integral across the two terms of $K_\Omega(x,y)\omega(y)$ in (2.20). ■

Applying the triangle inequality to this lemma gives us our initial estimate of the velocity gradient:

$$\|\nabla u(t,\cdot)\|_{L^\infty(\Omega)} \le \|\omega(t,\cdot)\|_{L^\infty(\Omega)} + \left\| \mathrm{p.\,v.}\int_\Omega \nabla_x K_\Omega(x,y)\omega(t,y)\,dy \right\|_{L^\infty(\Omega)} =: V(t). \quad (2.21)$$

2.7 Flow Map Estimates

We close this chapter by examining a key property of the flow map that ensures that an initially compactly supported vorticity remains compactly supported in Ω for all time. The defining property of the flow map given above in (2.11) can be written in integral form as

$$\eta(t,x) = x + \int_0^t u\left(s, \eta(s,x)\right)\,ds.$$

By subtracting two such expressions and using the triangle inequality, we have that

$$\begin{aligned}|\eta(t,x)-\eta(t,y)| &\le |x-y| + \int_0^t |u\left(s,\eta(s,x)\right) - u\left(s,\eta(s,y)\right)|\, ds \\ &= |x-y| + \int_0^t \frac{|u\left(s,\eta(s,x)\right) - u\left(s,\eta(s,y)\right)|}{|\eta(s,x)-\eta(s,y)|}\, |\eta(s,x)-\eta(s,y)|\, ds.\end{aligned}$$

By Lemma 2.2.5, this means

$$|\eta(t,x)-\eta(t,y)| \le |x-y| + \int_0^t C(\Omega)\, \|\nabla u\|_{L^\infty(\Omega)}\, |\eta(s,x)-\eta(s,y)|\, ds.$$

If we consider these expressions as functions of t with fixed x and y, Lemma 2.2.7 gives us the estimate

$$\begin{aligned}|\eta(t,x)-\eta(t,y)| &\le |x-y| e^{C\int_0^t \|\nabla u(s,\cdot)\|_{L^\infty}\, ds} \\ &\le |x-y| e^{C\int_0^t V(s)\, ds}.\end{aligned} \tag{2.22}$$

We note that if Ω is convex, then we could use Lemma 2.2.4 instead of Lemma 2.2.5 to have $C = 1$.

To obtain a lower bound, we have the following:

Lemma 2.7.1 *Let $x, y \in \Omega$. For all $t > 0$,*

$$|\eta(t,x)-\eta(t,y)| \ge (4e)^{1-e^{Ct}}\, |x-y|^{e^{Ct}}, \tag{2.23}$$

where $C = C(\omega_0)$.

Proof. Recall ([MB02, Lemma 8.1]) that u has a log-Lipschitz modulus of continuity μ_{LL} so that

$$|u(t,x) - u(t,y)| \le C_1 \mu_{LL}(|x-y|)$$

where

$$C_1 \leq C \left\| \omega(t,\cdot) \right\|_{L^\infty(\Omega)} \leq C \left\| \omega_0 \right\|_{L^\infty(\Omega)} = C(\omega_0)$$

and

$$\mu_{LL}(r) = \begin{cases} -r \ln r & \text{for } r \leq e^{-1}, \\ e^{-1} & \text{for } r \geq e^{-1}. \end{cases}$$

Since the diameter of the unit disk is 2, it is helpful to have a single expression for the modulus of continuity for $r \in [0,2]$. We define

$$\mu(r) = \begin{cases} -r \ln\left(re^{-1}/4\right) & \text{for } r \leq 2, \\ 2\ln(2e) & \text{for } r > 2. \end{cases}$$

Note that μ is non-decreasing and continuous, and that the constant 4 was chosen since it is twice the diameter of the disk. Since $\mu \geq \mu_{LL}$ for all r, u also admits μ as a modulus of continuity.

Now let $L(t) := |\eta(t,x) - \eta(t,y)|$. If $\eta(t,x) = \eta(t,y)$ then there is nothing to prove, so assume that $\eta(t,x) \neq \eta(t,y)$. Using the modulus of continuity, we can calculate that

$$\begin{aligned} L'(t) &= \frac{\eta(t,x) - \eta(t,y)}{|\eta(t,x) - \eta(t,y)|} \cdot \left(\partial_t \eta(t,x) - \partial_t \eta(t,y)\right) \\ &\geq - \left|\partial_t \eta(t,x) - \partial_t \eta(t,y)\right| \\ &= - \left|u(t,\eta(t,x)) - u(t,\eta(t,y))\right| \\ &\geq -C_1 \mu\left(|\eta(t,x) - \eta(t,y)|\right) \\ &= -C_1 \mu(L(t)). \end{aligned}$$

By Lemma A.6, this gives that

$$\int_{L(t)}^{L(0)} \frac{dr}{C_1 \mu(r)} \leq t. \tag{2.24}$$

Keeping in mind that $L(0) \leq 2$, we consider the integral

$$\begin{aligned}
\int_{L(t)}^{L(0)} \frac{dr}{\mu(r)} &= -\int_{L(t)}^{L(0)} \frac{dr}{r\ln(re^{-1}/4)} \\
&= \int_{L(0)}^{L(t)} \frac{dr}{r\ln(re^{-1}/4)} \\
&= \int_{\ln(L(0)e^{-1}/4)}^{\ln(L(t)e^{-1}/4)} \frac{ds}{s} \\
&= \ln \frac{\left|\ln(L(t)e^{-1}/4)\right|}{\left|\ln(L(0)e^{-1}/4)\right|}.
\end{aligned}$$

By (2.24), we can conclude that

$$\ln \frac{\left|\ln(L(t)e^{-1}/4)\right|}{\left|\ln(L(0)e^{-1}/4)\right|} \leq C_1 t.$$

This can be rearranged as

$$\left|\ln(L(t)e^{-1}/4)\right| \leq \left|\ln(L(0)e^{-1}/4)\right| e^{C_1 t}.$$

Since $|\ln s| = \left|\ln \frac{1}{s}\right|$, we have

$$\ln \frac{1}{L(t)e^{-1}/4} \leq \left(\ln \frac{1}{L(0)e^{-1}/4}\right) e^{C_1 t}$$

which can be exponentiated to give

$$\frac{4e}{L(t)} \leq \left(\frac{4e}{L(0)}\right)^{e^{C_1 t}}.$$

Finally, we can take the reciprocal of both sides to yield

$$L(t) \geq (4e)(4e)^{-e^{C_1 t}} \left(L(0)\right)^{e^{C_1 t}}$$

which is equivalent to

$$|\eta(t,x) - \eta(t,y)| \geq (4e)^{1-e^{C_1 t}} |x-y|^{e^{C_1 t}},$$

as desired. ■

The bounds 2.22 and 2.23 give us upper and lower bounds for how far apart two points can flow after time t. Let x be a point on the boundary of $\operatorname{supp}\omega(t,\cdot)$ and $y \in \partial\Omega$ such that $\operatorname{dist}(\operatorname{supp}\omega(t,\cdot), \partial\Omega) = |x-y|$. Because points on the boundary of Ω must stay on the boundary for all time and because the vorticity is passively transported by the flow, there exist $x' \in \partial(\operatorname{supp}\omega_0)$ and $y' \in \partial\Omega$ such that $x = \eta(t,x')$ and $y = \eta(t,y')$. By Lemma 2.7.1,

$$\begin{aligned} |x-y| &= \left|\eta(t,x') - \eta(t,y')\right| \\ &\geq (4e)^{1-e^{Ct}} \left|x'-y'\right|^{e^{Ct}} \\ &\geq (4e)^{1-e^{Ct}} \operatorname{dist}(\operatorname{supp}\omega_0, \partial\Omega)^{e^{Ct}} > 0, \end{aligned} \tag{2.25}$$

so that the distance from $\operatorname{supp}\omega(t,\cdot)$ to $\partial\Omega$ is bounded below away from zero for all time. This ensures that ω remains compactly supported for all time.

Chapter 3

Striated Regularity in a Bounded Domain

In this chapter, we will present the proof of Theorem 1.6.1. In [BK15], Theorem 1.5 is the analogue of Theorem 1.6.1 for the whole plane instead of a bounded domain. We will follow the outline of their proof, using their whole plane results whenever possible. The proof will proceed in several major steps as follows.

First, in Section 3.1, we present several lemmas from [BK15] that will be used and discuss how they can be applied to our bounded domain problem. In Section 3.2, we will deal with the issue of preparing the initial data to obtain approximate smooth solutions with compactly supported smooth vorticities. In Section 3.3, we will discuss some necessary transport equations and associated estimates for the sufficient family $\mathcal{Y}$ and various related quantities. In Section 3.4, we will prove a key estimate on the Hölder space norm of Y_n, defined in Section 3.3 as the pushforward of a member of $\mathcal{Y}$ under the flow maps corresponding to the smoothed initial data. In Section 3.5, we will obtain a bound on $\|\nabla u(t,\cdot)\|_{L^\infty(\Omega)}$ in terms of only itself (thus, an improvement on the initial estimate (2.21)) that will allow us to close the sequence of estimates in Section 3.6 with Grönwall's Lemmas (Lemmas 2.2.7 and 2.2.8), completing the proof of Theorem 1.6.1. We then close this chapter with Section 3.7 by looking more closely at Serfati's linear algebra lemma and how it is used to obtain the penultimate bound on the velocity gradient from Section 3.5.

For convenience, we will summarize the data given in the hypothesis of Theorem 1.6.1. For now, we take our domain Ω to be the open unit disk $B(0,1)$, only so that we can use the explicit Biot-Savart kernel K_Ω given by (2.18). In Section 3.6, we will discuss how to extend the result to an arbitrary open, bounded, and simply connected domain. We fix a time $T > 0$, where we have $t \in [0,T]$. We let $\mathcal{Y}_0$ be a $C^\alpha(\Omega)$ sufficient family of vector fields as defined by (1.9). We take an initial vorticity $\omega_0 \in L^\infty(\Omega)$ that is compactly supported in Ω, with an associated initial velocity $u_0 = K_\Omega[\omega_0]$ obtained via the Biot-Savart Law (Theorem 2.5.2). We assume that $\mathcal{Y}_0 \cdot \nabla u_0 \in C^\alpha(\Omega)$ and note that by Theorem 1.3 of [BK15], this assumption is equivalent to $\operatorname{div}(\omega_0 \mathcal{Y}_0) \in C^{\alpha-1}(\Omega)$. As in (2.19), for any compactly supported vorticity ω in Ω, we denote the extension by zero of ω to $\mathbb{R}^2$ by $\widetilde{\omega}$.

Lastly, we introduce the notation $C_T := C(T)$ for a constant that applies specifically to solutions to the Euler equations on $[0,T]$. In light of the estimate (2.25), any constant that depends on the distance from the support of $\omega(t,x)$ to the boundary of Ω may increase as T increases. This is because, while ω remains compactly supported for any finite time T, the distance from its support to the boundary of Ω could decrease in time, so the L^∞ norm of $K(x - y^*)$ and its gradient may increase in time. However, the estimates obtained will be valid on any arbitrary interval $[0, T]$, yielding global-in-time results.

3.1 Estimates Involving the Biot-Savart Kernel

In Section 2.6, we used the Biot-Savart Law (Theorem 2.5.2) to obtain bound (2.21), which involves the singular integral kernel ∇K. We will begin by presenting some basic estimates on kernels of this type that will be used in the proof of Theorem 1.6.1.

One of the most basic properties of the kernel K is that it is homogeneous of degree -1 (Proposition 2.1 of [MB02]):

$$|K(x)| \leq \frac{C}{|x|} \tag{3.1}$$

We now give a convenient expression for the difference of two values of the Biot-Savart kernel:

Lemma 3.1.1 *For nonzero x and y in $\mathbb{R}^2$, we have*

$$|K(x) - K(y)| = \frac{1}{2\pi} \frac{|x - y|}{|x|\,|y|}$$

Proof. If $x = (x_1, x_2)$, recall that $x^\perp = (-x_2, x_1)$. It is clear that $\left|x^\perp\right| = |x|$. First note that $2\pi\left[K(x) - K(y)\right] = \frac{x^\perp}{|x|^2} - \frac{y^\perp}{|y|^2}$. Taking the inner product of this vector with itself gives

$$\begin{aligned}
\left|\frac{x^\perp}{|x|^2} - \frac{y^\perp}{|y|^2}\right|^2 &= \left|\frac{|y|^2 x^\perp - |x|^2 y^\perp}{|x|^2 |y|^2}\right|^2 \\
&= \frac{|x|^2 |y|^4 - 2\,|x|^2 |y|^2 x^\perp \cdot y^\perp + |x|^4 |y|^2}{|x|^4 |y|^4} \\
&= |x|^2 |y|^2 \frac{|y|^2 - 2x^\perp \cdot y^\perp + |x|^2}{|x|^4 |y|^4} \\
&= \frac{\left|x^\perp - y^\perp\right|^2}{|x|^2 |y|^2}.
\end{aligned}$$

Since $\left|x^\perp - y^\perp\right| = \left|(x-y)^\perp\right| = |x-y|$, we can take a square root and see that

$$2\pi\left|K(x)-K(y)\right| = \left|\frac{x^\perp}{|x|^2} - \frac{y^\perp}{|y|^2}\right| = \frac{|x-y|}{|x|\,|y|},$$

proving the claim. ∎

The following estimate will be needed:

Proposition 3.1.2 *For a compactly supported $\omega \in L^\infty(\Omega)$ and $f \in C^\alpha(\Omega)$,*

$$\left\|\text{p.v.}\int_\Omega \nabla_x K_\Omega(x,y)\omega(y)\left[f(y)-f(x)\right]\,dy\right\|_{C^\alpha_x(\Omega)} \le C_T V(\omega)\left\|f\right\|_{C^\alpha(\Omega)},$$

where

$$V(\omega) := \left\|\omega\right\|_{L^\infty(\Omega)} + \left\|\text{p.v.}\int_\Omega \nabla_x K_\Omega(x-y)\omega(y)\,dy\right\|_{L^\infty(\Omega)}.$$

Proof. Let $\widetilde{f} := \mathcal{E}_H f$, where E_H is the Stein Hölder extension operator from Lemma 2.1.5, giving that $\left\|\widetilde{f}\right\|_{C^\alpha(\mathbb{R}^2)} \le C(\alpha)\left\|f\right\|_{C^\alpha(\Omega)}$. Recalling that $K_\Omega(x,y) = K(x-y) - K(x-y^*)$, we can use the compact support of ω to break up the Hölder norm as

$$\begin{aligned}
&\left\|\text{p.v.}\int_\Omega \nabla_x K_\Omega(x,y)\omega(y)\left[f(y)-f(x)\right]\,dy\right\|_{C^\alpha_x(\Omega)} \\
&\quad\le \left\|\text{p.v.}\int_{\mathbb{R}^2} \nabla_x K(x-y)\widetilde{\omega}(y)\left[\widetilde{f}(y)-\widetilde{f}(x)\right]\,dy\right\|_{C^\alpha_x(\mathbb{R}^2)} \\
&\qquad + \left\|\int_\Omega \nabla_x K(x-y^*)\omega(y)\left[f(y)-f(x)\right]\,dy\right\|_{C^\alpha_x(\Omega)} \\
&\quad:= I + J. \qquad (3.2)
\end{aligned}$$

By Lemmas 3.1, 3.2, and 3.3 of [BK15], I is bounded by

$$I \le CV(\omega)\left\|\widetilde{f}\right\|_{C^\alpha(\mathbb{R}^2)} \le CV(\omega)\left\|f\right\|_{C^\alpha(\Omega)}. \qquad (3.3)$$

Let $L(x,y) = \nabla_x K(x-y^*)\omega(y)$. We can further break up J as

$$\begin{aligned}
&\left\| \int_\Omega \nabla_x K(x-y^*)\omega(y) \left[f(y)-f(x)\right] \, dy \right\|_{C^\alpha_x(\Omega)} \\
&\quad = \left\| \int_\Omega L(x,y) \left[f(y)-f(x)\right] \, dy \right\|_{\dot{C}^\alpha_x(\Omega)} + \left\| \int_\Omega L(x,y) \left[f(y)-f(x)\right] \, dy \right\|_{L^\infty(\Omega)} \\
&\quad := J_1 + J_2. \qquad (3.4)
\end{aligned}$$

To estimate J_2, note that

$$\begin{aligned}
&\left| \int_\Omega \nabla_x K(x-y^*)\omega(y) \left[f(y)-f(x)\right] \, dy \right| \\
&\qquad \le \left(\sup_{x,y\in \operatorname{supp}(\omega)} \nabla_x K(x-y^*) \right) \|\omega\|_{L^\infty(\Omega)} \int_\Omega \frac{|f(y)-f(x)|}{|y-x|^\alpha} |y-x|^\alpha \, dy \\
&\qquad \le C_T \|\omega\|_{L^\infty(\Omega)} \|f\|_{C^\alpha(\Omega)} \int_\Omega |y-x|^\alpha \, dy \\
&\qquad = C(\Omega,\alpha,T) \|\omega\|_{L^\infty(\Omega)} \|f\|_{C^\alpha(\Omega)}. \qquad (3.5)
\end{aligned}$$

All that remains is to bound J_1. We will first need a bound on $\left\| \int_\Omega L(x,y)\, dy \right\|_{C^\alpha_x(\Omega)}$. Let $S := \operatorname{supp}(\omega)$ and recall that $\operatorname{dist}(S, \partial\Omega) > 0$. We have that

$$\begin{aligned}
\left\| \int_\Omega L(x,y)\, dy \right\|_{\dot{C}^\alpha_x(\Omega)} &= \sup_{x,x'\in\Omega} \frac{\left| \int_\Omega \nabla_x K(x-y^*)\omega(y)\, dy - \int_\Omega \nabla_x K(x'-y^*)\omega(y)\, dy \right|}{|x-x'|^\alpha} \\
&= \sup_{x,x'\in\Omega} \frac{\left| \int_S \nabla_x K(x-y^*)\omega(y)\, dy - \int_S \nabla_x K(x'-y^*)\omega(y)\, dy \right|}{|x-x'|^\alpha} \\
&\le \|\omega\|_{L^\infty(\Omega)} \sup_{x,x'\in\Omega} \int_S \frac{|\nabla_x K(x-y^*) - \nabla_x K(x'-y^*)|}{|x-x'|^\alpha} \, dy \\
&\le C(\Omega) \|\omega\|_{L^\infty(\Omega)} \sup_{y\in S} \|\nabla_x K(x-y^*)\|_{\dot{C}^\alpha_x(\Omega)} \\
&\le C_T \|\omega\|_{L^\infty(\Omega)}, \qquad (3.6)
\end{aligned}$$

where the last inequality follows because $\nabla_x K(x-y^*)$ is C^∞ on the closed set S, so is α-Hölder continuous on S. We note that the integral is switched to be over S to ensure that the singularity of $K(x-y^*)$ along $\partial\Omega$ is avoided. The supremum is then applied under the integral to give the integrand $\|\nabla_x K(x-y^*)\|_{\dot{C}^\alpha_x(\Omega)}$, of which we took the supremum over $y \in S$ to finish the estimate.

We also can calculate that

$$\begin{aligned}
\left\| \int_\Omega L(x,y)\,dy \right\|_{L^\infty(\Omega)} &= \left\| \int_\Omega \nabla_x K(x-y^*)\omega(y)\,dy \right\|_{L^\infty(\Omega)} \\
&= \left\| \int_S \nabla_x K(x-y^*)\omega(y)\,dy \right\|_{L^\infty(\Omega)} \\
&\le C(\Omega)\,\|\omega\|_{L^\infty(\Omega)}\,\|\nabla_x K(x-y^*)\|_{L^\infty(\Omega\times S)} \\
&\le C_T\,\|\omega\|_{L^\infty(\Omega)}\,, \qquad (3.7)
\end{aligned}$$

where the last inequality is due to the boundedness of $\nabla_x K(x-y^*)$ on S. Putting (3.6) and (3.7) together yields

$$\left\| \int_\Omega L(x,y)\,dy \right\|_{C^\alpha_x(\Omega)} \le C_T\,\|\omega\|_{L^\infty(\Omega)}\,. \qquad (3.8)$$

We now are ready to bound the term J_1 from (3.4). Using the definition of the Hölder norm and adding and subtracting the expression $L(x',y)f(x)$ in the second term that arises, we can calculate that

$$\begin{aligned}
J_1 &= \left\| \int_\Omega L(x,y)\left[f(y)-f(x)\right]\,dy \right\|_{\dot{C}^\alpha_x(\Omega)} \\
&= \sup_{x,x'\in\Omega} \frac{\left| \int_\Omega L(x,y)\left[f(y)-f(x)\right]\,dy - \int_\Omega L(x',y)\left[f(y)-f(x')\right]\,dy \right|}{|x-x'|^\alpha} \\
&\le \sup_{x,x'\in\Omega} \frac{\left| \int_\Omega \left[L(x,y)-L(x',y)\right] f(y)\,dy \right|}{|x-x'|^\alpha} + \sup_{x,x'\in\Omega} \frac{\left| \int_\Omega \left[L(x',y)f(x')-L(x,y)f(x)\right]\,dy \right|}{|x-x'|^\alpha} \\
&\le \left\| \int_\Omega L(x,y)\,dy \right\|_{\dot{C}^\alpha_x(\Omega)} \|f\|_{L^\infty(\Omega)} \\
&\qquad + \sup_{x,x'\in\Omega} \left| \int_\Omega \left[\frac{L(x',y)f(x')-L(x',y)f(x)}{|x-x'|^\alpha} + \frac{L(x',y)f(x)-L(x,y)f(x)}{|x-x'|^\alpha} \right] dy \right|
\end{aligned}$$

$$\le \left\| \int_\Omega L(x,y)\,dy \right\|_{\dot{C}^\alpha_x(\Omega)} \|f\|_{L^\infty(\Omega)}$$

$$+ \left\| \int_\Omega L(x,y)\,dy \right\|_{L^\infty(\Omega)} \|f\|_{\dot{C}^\alpha(\Omega)} + \left\| \int_\Omega L(x,y)\,dy \right\|_{\dot{C}^\alpha_x(\Omega)} \|f\|_{L^\infty(\Omega)}$$

$$= 2\left\| \int_\Omega L(x,y)\,dy \right\|_{\dot{C}^\alpha_x(\Omega)} \|f\|_{L^\infty(\Omega)} + \left\| \int_\Omega L(x,y)\,dy \right\|_{L^\infty(\Omega)} \|f\|_{\dot{C}^\alpha(\Omega)}$$

$$\le 2\left(\left\| \int_\Omega L(x,y)\,dy \right\|_{\dot{C}^\alpha_x(\Omega)} + \left\| \int_\Omega L(x,y)\,dy \right\|_{L^\infty(\Omega)} \right) \left(\|f\|_{\dot{C}^\alpha_x(\Omega)} + \|f\|_{L^\infty(\Omega)} \right)$$

$$= 2\left\| \int_\Omega L(x,y)\,dy \right\|_{C^\alpha_x(\Omega)} \|f\|_{C^\alpha_x(\Omega)}$$

$$\le C_T \|\omega\|_{L^\infty(\Omega)} \|f\|_{C^\alpha_x(\Omega)}, \tag{3.9}$$

where the last inequality is due to the bound (3.8). Using (3.2), we can now combine estimates (3.3), (3.5) and (3.9) to complete the proof. ∎

We are now ready to present an expression we will need in the proof along with a related estimate:

Proposition 3.1.3 *Let $\omega \in L^1(\Omega) \cap L^\infty(\Omega)$ be compactly supported and let Y be a vector field in $C^\alpha(\mathbb{R}^2)$. Then, for all $x \in \Omega$,*

$$\text{p. v.} \int \nabla_x K_\Omega(x,y) Y(y)\omega(y)\,dy = -\frac{\omega(x)}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(x) + \int_\Omega K(x-y)\,\text{div}\,(\omega Y)(y)\,dy$$

$$- \int_\Omega \nabla_x K(x-y^*) Y(y)\omega(y)\,dy. \tag{3.10}$$

Proof. Let $\widetilde{Y} = \mathcal{E}_H Y$. We start, as usual, by using the compact support of ω and expression (2.18) for K_Ω to break up the integral as

$$\text{p. v.} \int_\Omega \nabla_x K_\Omega(x,y) Y(y)\omega(y)\,dy$$

$$= \text{p. v.} \int_{\mathbb{R}^2} \nabla_x K(x-y)\widetilde{Y}(y)\widetilde{\omega}(y)\,dy - \int_\Omega \nabla_x K(x-y^*) Y(y)\omega(y)\,dy.$$

If we denote the first term of this expression by I, then by Proposition 4.2 of [BK15],

$$I = -\frac{\widetilde{\omega}(x)}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\widetilde{Y}(x) + \int_{\mathbb{R}^2} K(x-y)\,\mathrm{div}\,(\widetilde{\omega}Y)\,(y)\,dy$$

$$= -\frac{\omega(x)}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(x) + \int_{\Omega} K(x-y)\,\mathrm{div}\,(\omega Y)\,(y)\,dy,$$

which completes the proof. ■

Corollary 3.1.4 *Let $\omega \in L^1(\Omega) \cap L^\infty(\Omega)$ be compactly supported and let Y be a vector field in $C^\alpha(\mathbb{R}^2)$. Then, for all $x \in \Omega$,*

$$Y(x)\cdot\nabla u(x) = \int_\Omega K(x-y)\,\mathrm{div}(\omega Y)(y)\,dy - \int_\Omega \nabla_x K(x-y^*)Y(y)\omega(y)\,dy + \mathrm{p.\,v.}\int_\Omega \nabla_x K_\Omega(x,y)\,[Y(x)-Y(y)]\,\omega(y)\,dy \tag{3.11}$$

Proof. Let J represent the right-hand side of (3.11). Rearranging the terms of (3.10), adding $\mathrm{p.\,v.}\int_\Omega \nabla_x K_\Omega(x,y)\,[Y(x)-Y(y)]\,\omega(y)\,dy$ to both sides, and combining the left-hand side integrals gives that

$$\frac{\omega(x)}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(x) + \mathrm{p.\,v.}\int_\Omega \nabla_x K_\Omega(x,y)Y(x)\omega(y)\,dy = J.$$

Using Lemma 2.6.1, we can see that the left-hand side of this is equal to $Y(x)\cdot\nabla u(x)$, proving the statement. ■

3.2 Preparing the Initial Data and Approximate Smooth Solutions

In this section, we will prepare the initial data with mollification so we can work with smooth solutions. We will use the standard mollifiers given in Definition 2.1.3.

In general, we cannot use convolution with ω_0 since it is not defined outside of Ω. This issue can be resolved due to the compact support of ω_0 in Ω, and is in fact one of the primary reasons we include this hypothesis in Theorem 1.6.1. Since $\widetilde{\omega}_0$ is defined on $\mathbb{R}^2$, the convolution $\rho_n * \widetilde{\omega}_0$ is well-defined. Because $\operatorname{supp}(\omega_0) \subseteq \Omega$, there is a $\delta > 0$ satisfying $0 < \delta < \inf\{|x - z| : x \in \operatorname{supp}(\omega_0), z \in \partial\Omega\}$ (the supremum over all such δ is simply the distance from the support of ω_0 to the boundary of Ω). We choose N large enough so that $\frac{1}{N} < \frac{\delta}{2}$. Then, for all $n > N$ and $x \in \Omega$, we have that

$$\rho_n * \widetilde{\omega}_0(x) := \int_{\mathbb{R}^2} \rho_n(x - y)\widetilde{\omega}_0(y)\, dy = \int_{\Omega} \rho_n(x - y)\omega_0(y)\, dy.$$

This integral is well defined because $\operatorname{supp}(\rho_n(x - y)\omega_0(y))$ is contained in the closure of the set $\{z_1 + z_2 : z_1 \in \operatorname{supp}(\omega_0), |z_2| < \frac{\delta}{2}\}$, which is itself compactly supported in Ω. In a slight abuse of notation, we use the shorthand $\rho_n * \omega_0 := \rho_n * \widetilde{\omega}_0$.

Define $\omega_{0,n} := \rho_n * \omega_0$. By properties of convolution, for all $n > N$, $\omega_{0,n} \in C_c^\infty(\Omega)$, giving us a sequence of smooth vorticities that converge to ω_0 in $L^1(\Omega)$. For each $n > N$, we can obtain a unique smooth solution ([Wol33]) to the two-dimensional Euler equations (2.10) with $\omega_{0,n}$ as the smooth initial vorticity consisting of a smooth vorticity $\omega_n(t, x)$ with $\omega_n(0, x) = \omega_{0,n}(x)$ and a smooth velocity $u_n(t, x)$ recovered from $\omega_n(t, x)$ via the Biot-Savart

Law (Theorem 2.5.2) as

$$u_n(t,x) = \int_\Omega K_\Omega(x,y)\omega_n(t,y)\,dy.$$

We can obtain flow maps $\eta_n(t,x)$ given by (2.11) as the solution to the system

$$\begin{cases} \partial_t \eta_n(t,x) = u_n(t,\eta(t,x)), \\ \eta_n(0,x) = x. \end{cases}$$

Proposition 3.2.1 (Properties of ω_n and u_n) *For the above sequences ω_n and u_n, we have the following for all $n > N$:*

1. *$\omega_n \in C_c^\infty(\Omega)$ and $u_n \in C^\infty(\Omega)$.*

2. *For any n,*

   (a) $\|\omega_{0,n}\|_{L^\infty(\Omega)} \leq \|\omega_0\|_{L^\infty(\Omega)}$,

   (b) $\|\omega_{0,n}\|_{L^1(\Omega)} \leq \|\omega_0\|_{L^1(\Omega)}$, *and*

   (c) $\lim_{n\to\infty} \|\omega_{0,n} - \omega_0\|_{L^1(\Omega)} = 0$.

3. *ω_n and u_n are uniformly bounded, and*

$$\|u_n(t,\cdot)\|_{L^\infty(\Omega)} \leq C\,\|\omega_n(t,\cdot)\|_{L^\infty(\Omega)} \leq C\,\|\omega_0\|_{L^\infty(\Omega)}\,.$$

4. *There exists a function $\omega(t,\cdot) \in L^1(\Omega) \cap L^\infty(\Omega)$ such that, for any time,*

$$\omega_n(t,\cdot) \to \omega(t,\cdot) \quad \textit{in } L^1(\Omega).$$

5. *There exists a function $u = \int_\Omega K_\Omega(x,y)\omega(y)\,dy$ such that, for any time,*

$$u_n(t,\cdot) \to u(t,\cdot) \quad \textit{uniformly.}$$

6. *The velocity-vorticity pair (u, ω) is a weak solution to the vorticity-stream formulation of the two-dimensional Euler equations in Ω as defined in Definition 2.4.1.*

Proof. Property 1 follows from basic properties of convolution and the above discussion about the support of $\rho_n * \omega_0$. Property 2 is proved in section 8.2 of [MB02] (see Proposition 8.2). While the statements there are for the whole plane $\mathbb{R}^2$, they still apply here because of the compact support in Ω of $\omega_{0,n}$. The analogue of Property 3 in $\mathbb{R}^2$ is also proved in [MB02], but the statement there is no longer directly applicable since our velocities u_n are defined via Theorem 2.5.2 with K_Ω, not through Lemma 2.5.1 with K as in [MB02].

To prove Property 3, we first note that basic properties of mollifiers and the fact that vorticity is transported by the flow give that $\|\omega_n(t,\cdot)\|_{L^\infty(\Omega)} \leq \|\omega_0\|_{L^\infty(\Omega)}$. It remains to show that the velocities u_n are uniformly bounded. We take the radial cutoff function $a(x)$ as defined by (2.7). We can use the Biot-Savart Law (Theorem 2.5.2) and the compact support of ω_n to split u_n into several terms:

$$\begin{aligned}
u_n(t,x) &= \int_\Omega K_\Omega(x,y)\omega_n(t,y)\,dy \\
&= \int_{\mathbb{R}^2} K(x-y)\widetilde{\omega}_n(t,y)\,dy - \int_\Omega K(x-y^*)\omega_n(t,y)\,dy \\
&= \int_{\mathbb{R}^2} a(x-y)K(x-y)\widetilde{\omega}_n(t,y)\,dy + \int_{\mathbb{R}^2} (1-a(x-y))\,K(x-y)\widetilde{\omega}_n(t,y)\,dy \\
&\qquad\qquad - \int_\Omega K(x-y^*)\omega_n(t,y)\,dy.
\end{aligned}$$

Note that the first integral has a compactly supported integrand and that the second integral has cutoff the singularity of K so that its integrand is bounded. Recall that K is locally integrable and that (3.1) gave that $|K(x)| \leq C\,|x|^{-1}$. Furthermore, on the support of ω_n, $K(x-y^*)$ is smooth and bounded. Applying these observations and Hölder's inequality yields that

$$
\begin{aligned}
|u_n(t,x)| \le \|aK\|_{L^1(\mathbb{R}^2)} \|\omega_n(t,\cdot)\|_{L^\infty(\Omega)} + \|(1-a)K\|_{L^\infty(\mathbb{R}^2)} \|\omega_n(t,\cdot)\|_{L^1(\Omega)} \\
+ \int_\Omega |K(x-y^*)| \, |\omega_n(t,y)| \, dy \\
\le C \|\omega_n(t,\cdot)\|_{L^\infty(\Omega)} + C \|\omega_n(t,\cdot)\|_{L^1(\Omega)} + C_T \|\omega_n(t,\cdot)\|_{L^\infty(\Omega)} \\
\le C_T \|\omega_n(t,\cdot)\|_{L^1(\Omega)\cap L^\infty(\Omega)},
\end{aligned}
$$

proving Property 3.

Properties 4 to 6 follow exactly as in the proof of Proposition 8.2 of [MB02]. ■

According to Proposition 3.2.1, the sequences of smooth solutions ω_n and u_n converge to weak solutions ω and u that satisfy (2.10) and Theorem 2.5.2. We will use the smooth solutions ω_n and u_n for the majority of the proof (always assuming $n > N$) and pass to the limit as $n \to \infty$ in Section 3.6.

3.3 Transport Equations and Estimates

In this section, we will obtain some necessary transport equations for the pushforward of Y_0 under the flow maps η_n associated to the smooth solutions ω_n and u_n, as well as some estimates on the gradients of the flow maps. Let $\mathcal{Y}_0 = \left(Y_0^{(\lambda)}\right)_{\lambda\in\Lambda}$ be a sufficient $C^\alpha(\Omega)$ family of vector fields. Recall that a sufficient family was defined by (1.9) to be one such that

$$
\mathcal{Y} \in C^\alpha(\Omega), \quad \operatorname{div}\mathcal{Y} \in C^\alpha(\Omega), \quad \text{and } I_\Omega(\mathcal{Y}) > 0,
$$

where

$$
I_\Omega(\mathcal{Y}) := \inf_{x\in\Omega} \sup_{\lambda\in\Lambda} \left| Y^{(\lambda)} \right|.
$$

Let $Y_0 \in \mathcal{Y}_0$. As in (1.10), we define the pushforward of Y_0 under the flow maps η_n by

$$Y_n\left(t, \eta_n\left(t, x\right)\right) = \left(Y_0(x) \cdot \nabla\right) \eta_n\left(t, x\right). \tag{3.12}$$

Recall that an equivalent definition (as in (1.11)) is

$$Y_n(t, x) = \left(Y_0 \cdot \nabla \eta_n\right)\left(t, \eta_n^{-1}(t, x)\right). \tag{3.13}$$

We similarly define the pushforward of $\mathcal{Y}_0$ as in (2.6) by

$$\mathcal{Y}_n(t, \cdot) = \left(Y_n^{(\lambda)}(t, \cdot)\right)_{\lambda \in \Lambda}.$$

The equations and estimates in this section apply to any member of $\mathcal{Y}_0$, so we now turn our attention to an arbitrary element Y_0 and its pushforward Y_n. We will first need some transport equations.

Proposition 3.3.1 *Y_n satisifies the following:*

$$\partial_t Y_n + u_n \cdot \nabla Y_n = Y_n \cdot \nabla u_n \tag{3.14}$$

$$\partial_t \operatorname{div} Y_n + u_n \cdot \nabla \operatorname{div} Y_n = 0 \tag{3.15}$$

$$\partial_t \operatorname{div}\left(\omega_n Y_n\right) + u_n \cdot \nabla \operatorname{div}\left(\omega_n Y_n\right) = 0 \tag{3.16}$$

$$\operatorname{div}\left(Y_n(t, x)\right) = \operatorname{div} Y_0\left(\eta_n^{-1}(t, x)\right) \tag{3.17}$$

$$\frac{d}{dt} Y_n\left(t, \eta_n(t, x)\right) = \left(Y_n \cdot \nabla u_n\right)\left(t, \eta_n(t, x)\right) \tag{3.18}$$

$$\tag{3.19}$$

Proof. Taking the time derivative of the left-hand side of (3.12) gives

$$\begin{aligned}\partial_t\left(Y_n\left(t,\eta_n(t,x)\right)\right) &= \partial_t Y_n\left(t,\eta_n(t,x)\right) + \partial_j Y_n\left(t,\eta_n(t,x)\right)\partial_t \eta_n^j(t,x)\\ &= \partial_t Y_n\left(t,\eta_n(t,x)\right) + \partial_j Y_n\left(t,\eta_n(t,x)\right) u_n^j\left(t,\eta_n(t,x)\right)\\ &= \partial_t Y_n\left(t,\eta_n(t,x)\right) + (u_n\cdot\nabla)Y_n\left(t,\eta(t,x)\right), \end{aligned} \tag{3.20}$$

while the right-hand side time derivative of (3.12) is

$$\begin{aligned}\frac{d}{dt}\left(Y_0(x)\cdot\nabla\eta_n(t,x)\right) &= Y_0(x)\cdot\nabla\left(\partial_t\eta_n(t,x)\right)\\ &= Y_0(x)\cdot\nabla\left(u_n(t,\eta_n(t,x)\right)\\ &= Y_0(x)\cdot\left[\nabla u_n(t,\eta_n(t,x))\cdot\nabla\eta_n(t,x)\right]\\ &= \left(Y_0(x)\cdot\nabla\eta_n(t,x)\right)\cdot\nabla u_n(t,\eta_n(t,x))\\ &= Y_n\left(t,\eta_n(t,x)\right)\cdot\nabla u_n\left(t,\eta_n(t,x)\right).\end{aligned}$$

Setting the two sides equal to each other proves (3.14).

We now investigate the components of the time derivative of (3.12). Using (3.20), the i^{th} component is

$$\begin{aligned}&\partial_t Y_n^i(t,\eta_n(t,x)) + u_n(t,\eta_n(t,x))\cdot\nabla Y_n^i(t,\eta_n(t,x))\\ &\qquad= \partial_t\left[Y_0^j(x)\partial_j\eta_n^i(t,x)\right]\\ &\qquad= Y_0^j(x)\partial_j\partial_t\eta_n^i(t,x)\\ &\qquad= Y_0^j(x)\partial_j\left(u_n^i\left(t,\eta_n(t,x)\right)\right)\\ &\qquad= Y_0^j(x)\partial_k u_n^i\left(t,\eta_n(t,x)\right)\partial_j\eta_n^k(t,x).\end{aligned}$$

By letting $x' = \eta_n(t,x)$, so that $x = \eta_n^{-1}(t,x')$, we can express this as

$$\begin{aligned}\partial_t Y^i(t,x') + u_n(t,x') \cdot \nabla Y_n^i(t,x') &= Y_0^j\left(\eta_n^{-1}(t,x')\right)\partial_j \eta_n^k\left(t,\eta_n^{-1}(t,x')\right)\partial_k u_n^i(t,x') \\ &= Y_n^k(t,x')\partial_k u_n^i(t,x'),\end{aligned}$$

where we used (3.13) in the last line. We can now apply $\partial_{x_i'}$ to both sides and take the sum over $i = 1, 2$. This gives a left-hand side of

$$\begin{aligned}\partial_t \operatorname{div} Y_n + \partial_i\left(u_n^j \partial_j Y_n^i\right) &= \partial_t \operatorname{div} Y_n + \partial_i u_n^j \partial_j Y_n^i + u_n^j \partial_j \partial_i Y_n^i \\ &= \partial_t \operatorname{div} Y_n + \nabla u_n \cdot (\nabla Y_n)^T + u_n \cdot \nabla \operatorname{div} Y_n,\end{aligned}$$

while the right-hand side is

$$\begin{aligned}\partial_i\left(Y_n^k \partial_k u_n^i\right) &= \partial_i Y_n^k \partial_k u_n^i + Y_n^k \partial_k \partial_i u_n^i \\ &= \partial_i Y_n^k \partial_k u_n^i \\ &= \nabla u_n \cdot (\nabla Y_n)^T,\end{aligned}$$

where we used the fact that u_n is divergence-free in the second line. Comparing the left and right sides proves (3.15). The proof of (3.16) follows the same way as that of (3.15) and uses the fact that $\partial_t \omega_n + u_n \cdot \nabla \omega_n = 0$. Since (3.15) is equivalent to the quantity $\operatorname{div} Y_n$ being passively transported by the flow, this also immediately proves (3.17). Finally, we note that, in light of (3.20), we have proved (3.18) since it is simply an alternate way of expressing (3.14). ■

Some of our later calculations will involve $\operatorname{div}(\omega_n Y_n)$. The following regularity result for this quantity is a modification of Lemma 9.2 from [BK15]. To adapt the result to our bounded domain, we will need to use an extension of Y_n to the plane that will not be otherwise used, so we present those details in the appendix as Proposition A.7.

Proposition 3.3.2 *We have* $\operatorname{div}(\omega_n Y_n)(t,\cdot) \in C^{\alpha-1}(\Omega)$ *with*

$$\|\operatorname{div}(\omega_n Y_n)(t,\cdot)\|_{C^{\alpha-1}(\Omega)} \leq Ce^{C\int_0^t \|\nabla u_n(s,\cdot)\|_{L^\infty(\Omega)}\,ds}.$$

We next will obtain some estimates on the gradients of the flow map and its inverse. As in (2.21), we define

$$V_n(t) = \|\omega_n(t,\cdot)\|_{L^\infty(\Omega)} + \left\| \text{p. v.} \int_\Omega \nabla_x K_\Omega(x,y)\omega_n(t,y)\,dy \right\|_{L^\infty(\Omega)} \tag{3.21}$$

and recall that by the Biot-Savart Law (Theorem 2.5.2), we have $\|\nabla u_n(t,\cdot)\|_{L^\infty(\Omega)} \leq V_n(t)$, as in (2.21).

Lemma 3.3.3 *We have the following estimates:*

$$\|\nabla \eta_n(t,\cdot)\|_{L^\infty(\Omega)} \leq e^{\int_0^t V_n(s)\,ds}$$

$$\left\|\nabla \eta_n^{-1}(t,\cdot)\right\|_{L^\infty(\Omega)} \leq e^{\int_0^t V_n(s)\,ds}$$

Proof. Recall that the defining equation for η_n was given in (2.11):

$$\begin{cases} \partial_t \eta_n(t,x) = u_n(t,\eta_n(t,x)), \\ \eta_n(0,x) = x. \end{cases} \tag{3.22}$$

Integrating the differential equation in time over $[0,t]$ gives

$$\eta_n(t,x) = x + \int_0^t u_n\left(s,\eta_n(s,x)\right)\,ds.$$

Applying the gradient in the spatial variables and using the chain rule shows that

$$\nabla \eta_n(t,x) = I + \int_0^t \nabla u_n\left(s,\eta_n(s,x)\right)\cdot \nabla \eta_n(s,x)\,ds,$$

where I is the 2×2 identity matrix. Taking the L^∞ norm gives

$$\|\nabla \eta_n(t,\cdot)\|_{L^\infty(\Omega)} \leq 1 + \int_0^t \|\nabla u_n(s,\cdot)\|_{L^\infty(\Omega)} \|\nabla \eta_n(s,\cdot)\|_{L^\infty(\Omega)}\,ds.$$

We can now apply Lemma 2.2.7 to obtain

$$\|\nabla \eta_n(t,\cdot)\|_{L^\infty(\Omega)} \le e^{\int_0^t \|\nabla u_n(s)\|_{L^\infty(\Omega)}\,ds}.$$

By Lemma 2.6.1, this gives us that

$$\|\nabla \eta_n(t,\cdot)\|_{L^\infty(\Omega)} \le e^{\int_0^t V_n(s)\,ds}.$$

The bound for the inverse flow map $\nabla \eta_n^{-1}$ is more difficult because the flow is not autonomous. This estimate can be obtained by following the proof of Lemma 8.2 of [MB02] using $\nabla \eta_n^{-1}$ instead of their $X_\epsilon^{-t}(x)$. ■

3.4 Estimate of Y_n

In this section, we will begin putting the previous results together to work towards obtaining the bounds in Theorem 1.6.1. We start by bounding $\|Y_n(t,\cdot)\|_{C^\alpha(\Omega)}$, defined as the sum of the L^∞ norm and the $\dot{C}^\alpha$ norm. We begin with the L^∞ norm. Recall that by Lemma 3.3.1 we have

$$\frac{d}{dt} Y_n\left(t, \eta_n(t,x)\right) = \left(Y_n \cdot \nabla u_n\right)\left(t, \eta_n(t,x)\right).$$

By taking the inner product with $Y_n\left(t, \eta_n(t,x)\right)$ on both sides, we obtain

$$\frac{d}{dt} Y_n\left(t, \eta_n(t,x)\right) \cdot Y_n\left(t, \eta_n(t,x)\right) = \left(Y_n \cdot \nabla u_n\right)\left(t, \eta_n(t,x)\right) \cdot Y_n\left(t, \eta_n(t,x)\right). \tag{3.23}$$

Note that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left|Y_n\left(t, \eta_n(t,x)\right)\right|^2 &= \frac{1}{2}\sum_{i=1}^{2} \frac{d}{dt}\left[Y_n^i\left(t, \eta_n(t,x)\right)\right]^2 \\
&= Y_n\left(t, \eta_n(t,x)\right) \cdot \frac{d}{dt} Y_n\left(t, \eta_n(t,x)\right) \\
&= \left(Y_n \cdot \nabla u_n\right)\left(t, \eta_n(t,x)\right) \cdot Y_n\left(t, \eta_n(t,x)\right)
\end{aligned}$$

by (3.23). This gives that

$$\left| \frac{d}{dt} \left| Y_n \left(t, \eta_n(t,x)\right) \right|^2 \right| \leq 2 \left| Y_n(t, \eta_n(t,x)) \right| \left| \nabla u_n(t, \eta_n(t,x)) \right| \left| Y_n(t, \eta_n(t,x)) \right|$$

$$\leq 2 \left\| \nabla u_n \left(t, \eta_n(t, \cdot)\right) \right\|_{L^\infty} \left| Y_n \left(t, \eta_n(t,x)\right) \right|^2$$

$$= 2 \left\| \nabla u_n(t, \cdot) \right\|_{L^\infty} \left| Y_n \left(t, \eta_n(t,x)\right) \right|^2$$

$$\leq 2V_n(t) \left| Y_n \left(t, \eta_n(t,x)\right) \right|^2,$$

so that

$$\frac{d}{dt} \left| Y_n(t, \eta_n(t,x)) \right|^2 \leq 2V_n(t) \left| Y_n(t, \eta_n(t,x)) \right|^2 \tag{3.24}$$

and

$$\frac{d}{dt} \left| Y_n(t, \eta_n(t,x)) \right|^2 \geq -2V_n(t) \left| Y_n(t, \eta_n(t,x)) \right|^2. \tag{3.25}$$

We can now integrate (3.24) in time from 0 to t to see that

$$\int_0^t \frac{d}{ds} \left| Y_n(s, \eta_n(s,x)) \right|^2 \, ds \leq \int_0^t 2V_n(s) \left| Y_n(s, \eta_n(s,x)) \right|^2 \, ds.$$

Applying the Fundamental Theorem of Calculus to the left-hand sides gives that

$$\left| Y_n(t, \eta_n(t,x)) \right|^2 \leq \left| Y_n(0,x) \right|^2 + \int_0^t 2V_n(s) \left| Y_n(s, \eta_n(s,x)) \right|^2 \, ds.$$

Applying Lemma 2.2.7 and noting that $|Y_n(0,x)| = |Y_0(x)|$ shows that

$$\left| Y_n(t, \eta_n(t,x)) \right|^2 \leq \left| Y_0(x) \right|^2 e^{2 \int_0^t V_n(s) \, ds},$$

and applying Lemma 2.2.8 to (3.25) gives

$$\left| Y_n(t, \eta_n(t,x)) \right|^2 \geq \left| Y_0(x) \right|^2 e^{-2 \int_0^t V_n(s) \, ds}.$$

Taking square roots gives that

$$|Y_0(x)|\, e^{-\int_0^t V_n(s)\,ds} \leq |Y_n(t,\eta_n(t,x))| \leq |Y_0(x)|\, e^{\int_0^t V_n(s)\,ds}. \tag{3.26}$$

From this, we can conclude that

$$\|Y_n(t,\cdot)\|_{L^\infty(\Omega)} \leq \|Y_0\|_{L^\infty(\Omega)}\, e^{\int_0^t V_n(s)\,ds}. \tag{3.27}$$

We now turn to the quantity $\|Y_n(t,\cdot)\|_{\dot{C}^\alpha(\Omega)}$. We again begin with the transport equation given by Lemma 3.3.1:

$$\frac{d}{dt} Y_n\left(t,\eta_n(t,x)\right) = \left(Y_n \cdot \nabla u_n\right)\left(t,\eta_n(t,x)\right).$$

Integrating in time from 0 to t and applying the Fundamental Theorem of Calculus as above, we see that

$$Y_n(t,\eta_n(t,x)) = Y_0(x) + \int_0^t (Y_n \cdot \nabla u_n)(s,\eta_n(s,x))\,ds.$$

By re-expressing x as $\eta_n^{-1}(t,x')$, so that $\eta_n(t,x) = x'$, we can write this as

$$Y_n(t,x') = Y_0(\eta_n^{-1}(t,x')) + \int_0^t (Y_n \cdot \nabla u_n)(s,\eta_n(s,\eta_n^{-1}(t,x')))\,ds.$$

Taking the $\dot{C}^\alpha$ seminorm in Ω gives that

$$\|Y_n(t,\cdot)\|_{\dot{C}^\alpha(\Omega)} \leq \left\|Y_0(\eta_n^{-1}(t,x'))\right\|_{\dot{C}^\alpha(\Omega)} + \int_0^t \left\|(Y_n \cdot \nabla u_n)(s,\eta_n(s,\eta_n^{-1}(t,x')))\right\|_{\dot{C}^\alpha(\Omega)}\,ds.$$

By applying Lemma 2.2.6 to both of the right-hand side seminorms, we obtain the inequality

$$\begin{aligned} \|Y_n(t,\cdot)\|_{\dot{C}^\alpha(\Omega)} \leq{}& C\,\|Y_0\|_{\dot{C}^\alpha(\Omega)} \left\|\nabla \eta_n^{-1}(t,\cdot)\right\|_{L^\infty(\Omega)}^\alpha \\ &+ C\int_0^t \|(Y_n \cdot \nabla u_n)(s,\cdot)\|_{\dot{C}^\alpha(\Omega)} \left\|\nabla\left(\eta_n(s,\eta_n^{-1}(t,x'))\right)\right\|_{L^\infty(\Omega)}^\alpha\,ds. \end{aligned} \tag{3.28}$$

We next must estimate each of the factors appearing in (3.28). The first such estimate is that, by Lemma 3.3.3, we have

$$\left\|\nabla \eta_n^{-1}(t,\cdot)\right\|_{L^\infty(\Omega)}^{\alpha} \leq e^{\alpha \int_0^t V_n(s)\,ds}. \tag{3.29}$$

We will now focus on $\|(Y_n \cdot \nabla u_n)(s,\cdot)\|_{\dot{C}^\alpha(\Omega)}$, the first factor of the time integral in (3.28). By Corollary 3.1.4,

$$\begin{aligned}(Y_n \cdot \nabla u_n)(s,x) &= \int_\Omega K(x-y)\operatorname{div}(\omega_n Y_n)(s,y)\,dy - \int_\Omega \nabla_x K(x-y^*) Y_n(y)\omega_n(s,y)\,dy \\ &\qquad + \mathrm{p.\,v.}\int_\Omega \nabla_x K_\Omega(x,y)\left[Y_n(x) - Y_n(y)\right]\omega_n(s,y)\,dy \\ &=: \mathrm{I} - \mathrm{II} + \mathrm{III}. \end{aligned} \tag{3.30}$$

By Proposition 3.1.2,

$$\|\mathrm{III}\|_{C^\alpha(\Omega)} \leq C_T V_n(s)\,\|Y(s,\cdot)\|_{C^\alpha(\Omega)}. \tag{3.31}$$

Because ω_n is essentially bounded and compactly supported in Ω, and because $\nabla_x K(x-y^*)$ is smooth and bounded on the support of ω_n, we have that

$$\|\mathrm{II}\|_{C^\alpha(\Omega)} \leq C_T\,\|Y(s,\cdot)\|_{C^\alpha(\Omega)}. \tag{3.32}$$

To investigate $\|\mathrm{I}\|_{C^\alpha(\Omega)}$, we will need to extend Y_n to the whole plane. We will employ the Stein Hölder extension $\mathcal{E}_H$ from Lemma 2.1.5 to do so. Let $\widetilde{Y}_n = \mathcal{E}_H Y_n$. Note that the properties of $\mathcal{E}_H$ give that $\widetilde{Y}_n\big|_\Omega = Y_n$, $\widetilde{Y}_n \in C^\alpha(\mathbb{R}^2)$, and $\left\|\widetilde{Y}_n\right\|_{C^\alpha(\mathbb{R}^2)} \leq C(\alpha,\Omega)\,\|Y_n\|_{C^\alpha(\Omega)}$. Now let $U = \mathbb{R}^2 \setminus \operatorname{supp}\omega_n$, the complement of the support of ω_n. Recall that $\widetilde{\omega}_n$ is the extension by zero of ω_n to $\mathbb{R}^2$. Let $\phi \in C_c^\infty(U)$ be a test function on U. Note that

$$\int_U \operatorname{div}\left(\widetilde{\omega}_n \widetilde{Y}_n\right)(x)\phi(x)\,dx = -\int_U \widetilde{\omega}_n(x)\widetilde{Y}_n(x)\cdot\nabla\phi(x)\,dx = 0$$

since $\widetilde{\omega}_n$ is compactly supported on $\mathbb{R}^2 \backslash U$, so that, as a distribution, div $\left(\widetilde{\omega}\widetilde{Y}_n\right)$ is compactly supported in $\mathbb{R}^2 \setminus U = \operatorname{supp} \omega_n \subseteq \Omega$ ([FJ98, Section 1.4]). This means that

$$\begin{aligned}\int_\Omega K(x-y)\operatorname{div}(\omega_n Y_n)(s,y)\,dy &= \int_{\mathbb{R}^2} K(x-y)\operatorname{div}\left(\widetilde{\omega}_n\widetilde{Y}_n\right)(s,y)\,dy \\ &= \left[K * \operatorname{div}\left(\widetilde{\omega}_n\widetilde{Y}_n\right)\right](s,x). \end{aligned} \tag{3.33}$$

By Proposition 4.5 of [BK15], we have

$$\left\|K * \operatorname{div}\left(\widetilde{\omega}_n\widetilde{Y}_n\right)\right\|_{C^\alpha(\mathbb{R}^2)} \leq C\left(\left\|\widetilde{\omega}_n\widetilde{Y}_n\right\|_{L^1\cap L^\infty(\mathbb{R}^2)} + \left\|\operatorname{div}\left(\widetilde{\omega}_n\widetilde{Y}_n\right)\right\|_{C^{\alpha-1}(\mathbb{R}^2)}\right). \tag{3.34}$$

Using (3.33) and (3.34), we can see that

$$\begin{aligned}\|\mathrm{I}\|_{C^\alpha(\Omega)} &= \left\|\int_\Omega K(x-y)\operatorname{div}(\omega_n Y_n)(s,y)\,dy\right\|_{C^\alpha(\Omega)} \\ &= \left\|\left[K * \operatorname{div}\left(\widetilde{\omega}_n\widetilde{Y}_n\right)\right](s,\cdot)\right\|_{C^\alpha(\mathbb{R}^2)} \\ &\leq C\left(\left\|\left(\widetilde{\omega}_n\widetilde{Y}_n\right)(s,\cdot)\right\|_{L^1\cap L^\infty(\mathbb{R}^2)} + \left\|\operatorname{div}\left(\widetilde{\omega}_n\widetilde{Y}_n\right)(s,\cdot)\right\|_{C^{\alpha-1}(\mathbb{R}^2)}\right) \\ &\leq C_T\left\|\left(\omega_n Y_n\right)(s,\cdot)\right\|_{L^1\cap L^\infty(\Omega)} + C_T\left\|\operatorname{div}\left(\omega_n Y_n\right)(s,\cdot)\right\|_{C^{\alpha-1}(\Omega)},\end{aligned}$$

where the last line is justified because of the compact support in Ω of $\widetilde{\omega}_n$ and div $\left(\widetilde{\omega}_n\widetilde{Y}_n\right)$. We can now use Property 3 of Proposition 3.2.1, the bound given by (3.27), and Proposition 3.3.2 to obtain the estimate

$$\begin{aligned}\|\mathrm{I}\|_{C^\alpha(\Omega)} &\leq C(T,\Omega)\left\|\omega_n(s,\cdot)\right\|_{L^\infty(\Omega)}\left\|Y_n(s,\cdot)\right\|_{L^\infty(\Omega)} + C_T\left\|\operatorname{div}\left(\omega_n Y_n\right)(s,\cdot)\right\|_{C^{\alpha-1}(\Omega)} \\ &\leq C(T,\Omega,\omega_0)\left\|Y_0\right\|_{L^\infty(\Omega)} e^{\int_0^s V_n(\tau)\,d\tau} + C_T e^{C\int_0^s \|\nabla u_n(\tau,\cdot)\|_{L^\infty(\Omega)}\,d\tau} \\ &\leq C(T,\Omega,\omega_0,Y_0) e^{\int_0^s V_n(\tau)\,d\tau} + C_T e^{C\int_0^s V_n(\tau)\,d\tau} \\ &\leq C_T e^{C\int_0^s V_n(\tau)\,d\tau}.\end{aligned} \tag{3.35}$$

Putting estimates (3.31), (3.32) and (3.35) together with (3.30) gives us the estimate

$$\begin{aligned}\|(Y_n \cdot \nabla u_n)(s,\cdot)\|_{C^\alpha(\Omega)} &\leq \|\mathrm{I}\|_{C^\alpha(\Omega)} + \|\mathrm{II}\|_{C^\alpha(\Omega)} + \|\mathrm{III}\|_{C^\alpha(\Omega)} \\ &\leq C_T e^{C\int_0^s V_n(\tau)\,d\tau} + C_T \|Y_n(s,\cdot)\|_{C^\alpha(\Omega)} \\ &\quad + C_T V_n(s) \|Y_n(s,\cdot)\|_{C^\alpha(\Omega)} . \end{aligned} \tag{3.36}$$

We now turn to the quantity $\left\|\nabla\left(\eta_n(s,\eta_n^{-1}(t,x'))\right)\right\|_{L^\infty(\Omega)}^\alpha$, which is the second factor of the time integral in (3.28). We will follow the outline of the proof of Lemma 3.3.3 to bound this factor in a similar way. Note that the defining property of the flow maps given in (2.11) gives that

$$\partial_t \eta_n\left(\tau,\eta_n^{-1}(t,x')\right) = u_n\left(\tau,\eta_n\left(\tau,\eta_n^{-1}(t,x')\right)\right).$$

Taking the gradient with respect to the spatial variables and using the chain rule shows that

$$\partial_t \nabla\left(\eta_n\left(\tau,\eta_n^{-1}(t,x')\right)\right) = \nabla u_n\left(\tau,\eta_n\left(\tau,\eta_n^{-1}(t,x')\right)\right)\nabla\left(\eta_n\left(\tau,\eta_n^{-1}(t,x')\right)\right).$$

Integrating both sides in time from s to t and using the Fundamental Theorem of Calculus gives that

$$\begin{aligned}\nabla\left(\eta_n\left(\tau,\eta_n^{-1}(t,x')\right)\right)\big|_{\tau=t} &- \nabla\left(\eta_n\left(s,\eta_n^{-1}(t,x')\right)\right) \\ &= \int_s^t \nabla u_n\left(\tau,\eta_n\left(\tau,\eta_n^{-1}(t,x')\right)\right)\nabla\left(\eta_n\left(\tau,\eta_n^{-1}(t,x')\right)\right)\,d\tau.\end{aligned}$$

Because $\nabla\left(\eta_n\left(\tau,\eta_n^{-1}(t,x')\right)\right)\big|_{\tau=t} = I^{2\times 2}$ ([MB02, Section 1.3]), this can be rearranged to yield

$$\nabla\left(\eta_n\left(s,\eta_n^{-1}(t,x')\right)\right) = I^{2\times 2} - \int_s^t \nabla u_n\left(\tau,\eta_n\left(\tau,\eta_n^{-1}(t,x')\right)\right)\nabla\left(\eta_n\left(\tau,\eta_n^{-1}(t,x')\right)\right)\,d\tau.$$

Taking the L^∞ norm gives that

$$\left\|\nabla\left(\eta_n\left(s,\eta_n^{-1}(t,x')\right)\right)\right\|_{L^\infty(\Omega)} \le 1+\int_s^t \|\nabla u_n(\tau,\cdot)\|_{L^\infty(\Omega)} \left\|\nabla\left(\eta_n\left(s,\eta_n^{-1}(t,x')\right)\right)\right\|_{L^\infty(\Omega)}.$$

Applying Lemma 2.2.7 results in the estimate

$$\left\|\nabla\left(\eta_n\left(s,\eta_n^{-1}(t,x')\right)\right)\right\|_{L^\infty(\Omega)} \le e^{\int_s^t \|\nabla u_n(\tau,\cdot)\|_{L^\infty(\Omega)}\,d\tau} \le e^{\int_s^t V_n(\tau)\,d\tau}.$$

Thus, we can bound the factor in (3.28) with

$$\left\|\nabla\left(\eta_n\left(s,\eta_n^{-1}(t,x')\right)\right)\right\|_{L^\infty(\Omega)}^{\alpha} \le e^{\alpha\int_s^t V_n(\tau)\,d\tau}. \tag{3.37}$$

We are now ready to begin the task of putting the various estimates obtained so far together to obtain a bound on $\|Y_n(t,\cdot)\|_{C^\alpha(\Omega)}$. For notational clarity, we will suppress the domain Ω of the norms and denote quantities such as $\|f(t,\cdot)\|_X$ as simply $\|f(t)\|_X$. By (3.27) and from applying (3.29), (3.36) and (3.37) to (3.28), we have

$$\begin{aligned}
\|Y_n(t)\|_{C^\alpha} &= \|Y_n(t)\|_{L^\infty} + \|Y_n(t)\|_{\dot{C}^\alpha} \\
&\le \|Y_0\|_{L^\infty} e^{\int_0^t V_n(\tau)\,d\tau} + C\|Y_0\|_{\dot{C}^\alpha} e^{\alpha\int_0^t V_n(\tau)\,d\tau} \\
&\qquad + C_T \int_0^t \left[e^{C\int_0^s V_n(\tau)\,d\tau} + \|Y_n(s)\|_{C^\alpha} + V_n(s)\|Y_n(s)\|_{C^\alpha}\right] e^{\alpha\int_s^t V_n(\tau)\,d\tau}\,ds.
\end{aligned}$$

Because $\alpha < 1$ and $V_n > 0$, we can omit the α coefficients, combine the first two terms, and simplify the integrand to see that

$$\begin{aligned}
\|Y_n(t)\|_{C^\alpha} &\le C\|Y_0\|_{C^\alpha} e^{\int_0^t V_n(\tau)\,d\tau} + C_T\int_0^t e^{C\int_0^t V_n(\tau)\,d\tau}\,ds \\
&\qquad + C_T\int_0^t \left[\|Y_n(s)\|_{C^\alpha} + V_n(s)\|Y_n(s)\|_{C^\alpha}\right] e^{\int_s^t V_n(\tau)\,d\tau}\,ds \\
&\le \left[C\|Y_0\|_{C^\alpha} + C_T t\right] e^{C\int_0^t V_n(\tau)\,d\tau} + \int_0^t C_T\left(1+V_n(s)\right)\|Y_n(s)\|_{C^\alpha} e^{C\int_s^t V_n(\tau)\,d\tau}\,ds.
\end{aligned}$$

Multiplying both sides by $e^{-C\int_0^t V_n(\tau)\,d\tau}$ gives that

$$\|Y_n(t)\|_{C^\alpha} e^{-C\int_0^t V_n(\tau)\,d\tau} \le C\|Y_0\|_{C^\alpha} + C_T t + \int_0^t C_T(1+V_n(s))\|Y_n(s)\|_{C^\alpha} e^{-C\int_0^s V_n(\tau)\,d\tau}\,ds.$$

If we let $\xi_n(t) := \|Y_n(t)\|_{C^\alpha} e^{-C\int_0^t V_n(\tau)\,d\tau}$, then we have

$$\xi_n(t) \le C\|Y_0\|_{C^\alpha} + C_T t + \int_0^t C_T(1+V_n(s))\xi_n(s)\,ds,$$

so we can apply Lemma 2.2.7 to find that

$$\xi_n(t) \le [C\|Y_0\|_{C^\alpha} + C_T t]\, e^{\int_0^t C_T(1+V_n(s))\,ds} \le C_T(1+t)e^{\int_0^t C_T(1+V_n(s))\,ds}.$$

Thus, we obtain the estimate

$$\|Y_n(t)\|_{C^\alpha(\Omega)} \le C_T(1+t)e^{\int_0^t [C_T + C_T V_n(s)]\,ds}. \tag{3.38}$$

3.5 Improved Estimate of the Velocity Gradient

Recall that $V_n(t)$ was defined by (3.21) as

$$V_n(t) = \|\omega_n(t,\cdot)\|_{L^\infty(\Omega)} + \left\|\mathrm{p.\,v.}\int_\Omega \nabla_x K_\Omega(x,y)\omega_n(t,y)\,dy\right\|_{L^\infty(\Omega)} \tag{3.39}$$

and that, as in (2.21), $\|\nabla u_n(t)\|_{L^\infty(\Omega)} \le V_n(t)$.

We now need to bound $V_n(t)$ in terms of $\|Y_n(t)\|_{C^\alpha(\Omega)}$ so that, with (3.38), we will have $V_n(t)$ bounded in terms of itself, which will allow us to close the estimates with Grönwall's Inequality to obtain the final bound on $\|\nabla u_n(t)\|_{L^\infty(\Omega)}$ that will give us the estimates in Theorem 1.6.1. This is the climax of the proof, and indeed is where Serfati made use of his linear algebra lemma and where the various approaches of Serfati, Bertozzi and Constantin, and Chemin differ most significantly. We will use the approach taken in

Section 10.3 of [BK15], where the details are carried out for the full plane case, noting where and how adjustments need to be made to account for the extra term in the Biot-Savart kernel K_Ω in our domain. The fact that the extra term $K(x - y^*)$ is smooth and bounded on the support of $\omega(t, y)$ means that the results will largely carry through without significant differences.

We start by fixing $t \in [0, T]$ and $x \in \Omega$. First, note that (3.39) along with Property 3 of Proposition 3.2.1 gives that

$$V_n(t) \leq C(\omega_0) + \left\| \text{p. v.} \int_\Omega \nabla_x K_\Omega(x, y)\omega_n(t, y)\, dy \right\|_{L^\infty}.$$

Using the cutoff function a_r defined in (2.7), we can let $r \in (0, 1)$ and split the integral term as

$$\begin{aligned}
&\text{p. v.} \int_\Omega \nabla_x K_\Omega(x, y)\omega_n(t, y)\, dy \\
&\qquad = \text{p. v.} \int_\Omega \nabla(a_r K)(x - y)\omega_n(t, y)\, dy \\
&\qquad\qquad + \text{p. v.} \int_\Omega \nabla\left((1 - a_r)K\right)(x - y)\omega_n(t, y)\, dy \\
&\qquad\qquad - \int_\Omega \nabla K(x - y^*)\omega_n(t, y)\, dy \\
&\qquad = \text{p. v.} \int_{\mathbb{R}^2} \nabla(a_r K)(x - y)\widetilde{\omega}_n(t, y)\, dy \\
&\qquad\qquad + \text{p. v.} \int_{\mathbb{R}^2} \nabla\left((1 - a_r)K\right)(x - y)\widetilde{\omega}_n(t, y)\, dy \\
&\qquad\qquad - \int_\Omega \nabla K(x - y^*)\omega_n(t, y)\, dy \\
&\qquad =: \mathrm{I} + \mathrm{II} - \mathrm{III}.
\end{aligned} \tag{3.40}$$

Note that, since $K(x - y^*)$ is smooth and bounded on the support of ω_n, we have

$$|\mathrm{III}| \leq C(T, \Omega, \omega_0) \tag{3.41}$$

by Proposition 3.2.1.

Since $\nabla(1 - a_r) = -\nabla a_r$, due to the properties of a_r described in Section 2.1, we have that $\nabla(1 - a_r)$ is supported on $[r, 2r]$. This means that, on the support of $\nabla(1 - a_r)$, we have $|x - y| \leq 2r$. Since K is radially symmetric and and decreasing in $|x - y|$, and because a_r is smooth, we have

$$\begin{aligned}
|\nabla((1 - a_r)K)| &\leq |(1 - a_r)\nabla K| + |\nabla a_r \otimes K| \\
&\leq |\nabla K| + C\,|K| \\
&\leq \frac{C}{|x - y|^2}.
\end{aligned}$$

Using Proposition 3.2.1 several times, this gives that

$$\begin{aligned}
|\mathrm{II}| &\leq C \int_{B^c(x,r)} \frac{|\widetilde{\omega}_n(t,y)|}{|x - y|^2}\, dy \\
&= C \int_{B(x,1)\setminus B^c(x,r)} \frac{|\widetilde{\omega}_n(t,y)|}{|x - y|^2}\, dy + C \int_{B^c(x,1)} \frac{|\widetilde{\omega}_n(t,y)|}{|x - y|^2}\, dy \\
&\leq C\, \|\omega_0\|_{L^\infty(\Omega)} \int_0^{2\pi} \int_r^1 \frac{1}{\rho^2} \rho\, d\rho d\theta + C \left\| \frac{1}{|x - \cdot|^2} \right\|_{L^\infty(B^c(x,1))} \int_{B^c(x,1)} |\widetilde{\omega}_n(t,y)|\, dy \\
&\leq 2\pi C\, \|\omega_0\|_{L^\infty(\Omega)} (-\ln r) + C(1)\, \|\omega_n(t,\cdot)\|_{L^1(\Omega)} \\
&\leq C\,(1 - \ln r)\, \|\omega_0\|_{(L^1 \cap L^\infty)(\Omega)}
\end{aligned}$$

so that

$$|\mathrm{II}| \leq C(1 - \ln r). \tag{3.42}$$

In order to estimate $|\mathrm{I}|$, we first choose $Y_0 \in \mathcal{Y}_0$ such that

$$\left|Y_0\left(\eta_n^{-1}(t,x)\right)\right| \geq I_\Omega(\mathcal{Y}_0). \tag{3.43}$$

Note that, because of (3.26), this means that

$$|Y_n(t,x)| \geq I_\Omega(\mathcal{Y}_0)e^{-\int_0^t V_n(s)\,ds}. \tag{3.44}$$

In order to apply full plane results, we will need to extend the vector fields Y_0 and Y_n to the whole plane in a way that preserves the pushforward identity. As outlined in Section 2.5, there exist stream functions $\psi_n(t,x) \in C^\infty(\Omega)$ given by (2.15) so that $u_n = \nabla^\perp \psi_n$. We now will use the Stein extension operator $\mathcal{E}$ for Ω and its properties given by Lemma 2.1.4. Define $\psi_n^* := \mathcal{E}\psi_n$. Since $\psi_n \in W^{2,\infty}(\Omega)$, we have that $\psi_n^* \in W^{2,\infty}(\mathbb{R}^2)$. We then define $u_n^* := \nabla^\perp \psi_n^*$, which gives that $u_n^* \in W^{1,\infty}(\mathbb{R}^2)$. By construction, we have $\operatorname{div} u_n^* = \operatorname{div} \nabla^\perp(\mathcal{E}\psi_n) = 0$, so that u_n^* is divergence-free. Corresponding to the velocities u_n^*, we have flow maps η_n^*, obtained by solving (2.11). We note that, even though u_n^* and η_n^* are not obtained by applying $\mathcal{E}$ to u_n and η_n, respectively, we still have that $u_n^*\big|_\Omega = u_n$ and $\eta_n^*\big|_\Omega = \eta_n$. As in the previous section, we apply the Stein Hölder extension $\mathcal{E}_H$ from Lemma 2.1.5 to define $\widetilde{Y}_0 := \mathcal{E}_H Y_0$. Note that $\widetilde{Y}_0 \in C^\alpha(\mathbb{R}^2)$ and $\widetilde{Y}_0\big|_\Omega = Y_0$. We then pushforward $\widetilde{Y}_0$ according to (1.11) as

$$\widetilde{Y}_n(t,x) = \left(\widetilde{Y}_0 \cdot \nabla \eta_n^*\right)\left(t, (\eta_n^*)^{-1}(t,x)\right) \tag{3.45}$$

and note that we have $\widetilde{Y}_n\big|_\Omega = Y_n$.

Since $\mathrm{I} = \mathrm{p.v.}\int_{\mathbb{R}^2} \nabla(a_r K)(x-y)\widetilde{\omega}_n(t,y)\,dy$ is an integral over the whole plane and since $\widetilde{\omega}_n \in C^\infty(\mathbb{R}^2)$, $\widetilde{\omega}_n \in (L^1 \cap L^\infty)(\mathbb{R}^2)$, and $\widetilde{Y}_0 \in C^\alpha(\mathbb{R}^2)$, using (3.44), the lengthy calculations in Section 10.3 of [BK15], including the application of Serfati's linear algebra lemma (see Section 3.7 for a discussion of how the lemma is used), apply directly with $\widetilde{\omega}_n$ used in place of their ω_ϵ and $\widetilde{Y}_0$ used in place of their Y_0 to obtain the bound

$$\begin{aligned}
\sup_{x\in\mathbb{R}^2} |\mathrm{I}| &\leq C(\alpha) \left\| \widetilde{Y}_0 \right\|_{L^\infty(\mathbb{R}^2)} e^{8\int_0^t V_n(s)\, ds} \\
&\qquad \left(\left\| \widetilde{Y}_n \right\|_{C^\alpha(\mathbb{R}^2)} \left\| \widetilde{\omega}_0 \right\|_{L^\infty(\mathbb{R}^2)} + \left\| \operatorname{div}\left(\widetilde{\omega}_n \widetilde{Y}_n\right) \right\|_{C^{\alpha-1}(\mathbb{R}^2)} \right) r^\alpha \\
&\qquad + \left\| \widetilde{\omega}_0 \right\|_{L^\infty(\mathbb{R}^2)} \qquad (3.46) \\
&\leq C \left\| Y_0 \right\|_{L^\infty(\Omega)} e^{8\int_0^t V_n(s)\, ds} \left(\left\| Y_n(t) \right\|_{C^\alpha(\Omega)} \left\| \omega_0 \right\|_{L^\infty(\Omega)} + \left\| \operatorname{div}\left(\omega_n Y_n\right)(t) \right\|_{C^{\alpha-1}(\Omega)} \right) r^\alpha \\
&\qquad + \left\| \omega_0 \right\|_{L^\infty(\Omega)} . \qquad (3.47)
\end{aligned}$$

Using (3.38) and Proposition 3.3.2 and with $C = C(\omega_0, Y_0, \alpha, \Omega, T)$, this gives that

$$\begin{aligned}
\sup_{x\in\mathbb{R}^2} |\mathrm{I}| &\leq C e^{8\int_0^t V_n(s)\, ds} \left(C(1+t) e^{\int_0^t [C + CV_n(s)]\, ds} + C e^{C\int_0^t \|\nabla u_n(s,\cdot)\|_{L^\infty(\Omega)}\, ds} \right) r^\alpha + C \\
&\leq \left[C(1+t) e^{\int_0^t [C + CV_n(s)]\, ds} + C e^{\int_0^t CV_n(s)\, ds} \right] r^\alpha + C. \\
&\leq C(1+t) e^{\int_0^t [C + CV_n(s)]\, ds} r^\alpha + C. \qquad (3.48)
\end{aligned}$$

We can now combine (3.41), (3.42) and (3.48) with (3.40) to find that

$$\begin{aligned}
V_n(t) &\leq C(1+t) e^{\int_0^t [C + CV_n(s)]\, ds} r^\alpha + C + C(1 - \ln r) + C \\
&\leq C(1+t) e^{\int_0^t [C + CV_n(s)]\, ds} r^\alpha + C(1 - \ln r). \qquad (3.49)
\end{aligned}$$

3.6 Closing the Estimates and Generalizing the Domain

We now fix a constant C_0 satisfying (3.49), so that

$$V_n(t) \leq C_0(1+t) e^{C_0 \int_0^t (1 + V_n(s))\, ds} r^\alpha + C_0(1 - \ln r).$$

We set $r = e^{-\frac{C_0}{\alpha}\int_0^t (1+V_n(s))\,ds}$, so that

$$r^\alpha = e^{-C_0\int_0^t (1+V_n(s))\,ds}$$

and

$$1 - \ln r = 1 + C_1 \int_0^t (1 + V_n(s))\,ds,$$

with $C_1 = \frac{C_0}{\alpha}$. This value of r gives that

$$\begin{aligned} V_n(t) &\le C_0(1+t) + C_0\left(1 + C_1\int_0^t (1+V_n(s))\,ds\right) \\ &\le C(1+t) + C\int_0^t (1+V_n(s))\,ds \\ &\le C(1+t) + C\int_0^t V_n(s)\,ds. \end{aligned}$$

Using Lemma 2.2.7 allows us to conclude that

$$\begin{aligned} V_n(t) &\le C(1+t)e^{\int_0^t C\,ds} \\ &\le Ce^{\ln(1+t)}e^{Ct} \\ &\le Ce^{Ct}, \end{aligned}$$

since $\ln(1+t) \le t$. This gives us the estimate

$$\|\nabla u_n(t,\cdot)\|_{L^\infty(\Omega)} \le V_n(t) \le Ce^{Ct}. \tag{3.50}$$

We can now use (3.50) to close all of the previous estimates we have obtained. Applying it to (3.38) shows that

$$\begin{aligned} \|Y_n(t,\cdot)\|_{C^\alpha(\Omega)} &\le C(1+t)e^{\int_0^t [C+Ce^{Cs}]\,ds} \\ &\le Ce^{\ln(1+t)+Ct+Ce^{Ct}} \\ &\le Ce^{Ce^{Ct}}, \end{aligned} \tag{3.51}$$

since $\ln(1+t)$ and Ct are both bounded above by e^{Ct} for $t \geq 0$. Using (3.50) and (3.51) with (3.36) gives the estimate

$$\begin{aligned}
\|Y_n \cdot \nabla u_n(t,\cdot)\|_{C^\alpha(\Omega)} &\leq Ce^{\int_0^t Ce^{Cs}\,ds} + Ce^{Ce^{Ct}} + Ce^{Ct}e^{Ce^{Ct}} \\
&\leq Ce^{Ce^{Ct}} + Ce^{Ct+Ce^{Ct}} \\
&\leq Ce^{Ce^{Ct}}, \qquad (3.52)
\end{aligned}$$

where we used the fact that $t \leq e^{Ct}$. Using (3.50) with Proposition 3.3.2 gives that

$$\|\mathrm{div}\,(\omega_n Y_n)\,(t,\cdot)\|_{C^{\alpha-1}(\Omega)} \leq Ce^{\int_0^t Ce^{Cs}\,ds} \leq Ce^{Ce^{Ct}}. \qquad (3.53)$$

Using (3.50) with Lemma 3.3.3 gives

$$\|\nabla \eta_n(t,\cdot)\|_{L^\infty(\Omega)} \leq e^{\int_0^t Ce^{Cs}\,ds} \leq Ce^{Ce^{Ct}}. \qquad (3.54)$$

Using (3.50) with (3.26) gives that

$$|Y_n\,(t,\eta_n(t,x))| \geq |Y_0(x)|\,e^{-\int_0^t Ce^{Cs}\,ds} \geq |Y_0(x)|\,e^{-Ce^{Ct}}. \qquad (3.55)$$

We can use (3.17) from Lemma 3.3.1 along with Lemma 2.2.6, the fact that Ω is bounded, Lemma 3.3.3, and (3.50) to see that

$$\begin{aligned}
\|\mathrm{div}\,Y_n(t,\cdot)\|_{C^\alpha(\Omega)} &= \left\|\mathrm{div}\,Y_0\left(\eta_n^{-1}(t,\cdot)\right)\right\|_{C^\alpha(\Omega)} \\
&= \left\|\mathrm{div}\,Y_0\left(\eta_n^{-1}(t,\cdot)\right)\right\|_{L^\infty(\Omega)} + \left\|\mathrm{div}\,Y_0\left(\eta_n^{-1}(t,\cdot)\right)\right\|_{\dot{C}^\alpha(\Omega)} \\
&\leq \|\mathrm{div}\,Y_0\|_{L^\infty(\Omega)} \\
&\qquad + C\,\|\mathrm{div}\,Y_0\|_{\dot{C}^\alpha(\Omega)}\left[\left\|\eta_n^{-1}(t,\cdot)\right\|_{L^\infty(\Omega)} + \left\|\nabla\eta_n^{-1}(t,\cdot)\right\|_{L^\infty(\Omega)}\right]^\alpha \\
&\leq \|\mathrm{div}\,Y_0\|_{L^\infty(\Omega)} + C\,\|\mathrm{div}\,Y_0\|_{\dot{C}^\alpha(\Omega)}\left[C(\Omega) + e^{\int_0^t V_n(s)\,ds}\right]^\alpha \\
&\leq \|\mathrm{div}\,Y_0\|_{L^\infty(\Omega)} + C\,\|\mathrm{div}\,Y_0\|_{\dot{C}^\alpha(\Omega)}\left[(C+1)\,e^{\int_0^t V_n(s)\,ds}\right]^\alpha
\end{aligned}$$

$$\leq \|\operatorname{div} Y_0\|_{L^\infty(\Omega)} + C\,(C+1)^\alpha \,\|\operatorname{div} Y_0\|_{\dot{C}^\alpha(\Omega)}\, e^{\alpha \int_0^t V_n(s)\,ds}$$

$$\leq C\,\|\operatorname{div} Y_0\|_{C^\alpha(\Omega)}\, e^{\alpha \int_0^t C e^{Cs}\,ds}$$

$$\leq C\,\|\operatorname{div} Y_0\|_{C^\alpha(\Omega)}\, e^{Ce^t}. \tag{3.56}$$

The calculations verifying that these estimates hold in the limit as $n \to \infty$ are identical to those in Section 10.4 of [BK15] (where they used the mollification parameter ϵ and let $\epsilon \to 0$ rather than using $1/n$ and letting $n \to \infty$), so are not repeated here. We note that these calculations themselves are adaptations of Chemin's arguments in [Che91, Che93]. We can take the supremum over all $Y_0 \in \mathcal{Y}_0$ so that (3.50) gives the bound (1.12), (3.51) gives the bound (1.13), (3.56) gives the bound (1.14), (3.53) gives the bound (1.15), (3.52) gives the bound (1.16), (3.54) gives the bound (1.17), and (3.55) gives the bound (1.18), proving Theorem 1.6.1 for $\Omega = B(0,1)$.

Though our proof has been specific to the unit disk $\Omega = B(0,1)$, we note that we never used any properties specific to the unit disk (e.g., convexity) in our proof. Let Ω be an arbitrary simply connected bounded domain in the plane. As in Section 2.5, the Biot-Savart kernel for Ω is

$$K_\Omega(x,y) := \nabla_x^\perp G_\Omega(x,y),$$

where G_Ω is Green's function for Ω. As in [Eva10, Section 2.2], we can write this as

$$G_\Omega = G - \phi,$$

where $G(x,y) = \frac{1}{2\pi}\ln|y-x|$ is the fundamental solution to the Laplacian in $\mathbb{R}^2$ and ϕ is a harmonic solution to the boundary-value problem

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega \\ \phi = G & \text{on } \partial\Omega. \end{cases} \tag{3.57}$$

Thus, the Biot-Savart kernel for Ω is

$$\begin{aligned} K_\Omega(x,y) &= \nabla_x^\perp G_\Omega(x,y) \\ &= \nabla_x^\perp G(x,y) - \nabla_x^\perp \phi(x,y) \\ &= K(x-y) - \nabla_x^\perp \phi(x,y) \\ &= K(x-y) - R(x,y), \end{aligned} \tag{3.58}$$

where K is the Biot-Savart kernel for the plane as in Lemma 2.5.1 and $R(x,y) := \nabla_x^\perp \phi(x,y)$. Following arguments identical to those in Section 2.5, we have the following Biot-Savart Law for Ω:

Theorem 3.6.1 (Biot-Savart Law in Ω) *Let $u(t,x)$ be the divergence-free velocity associated with the vorticity $\omega(t,x) \in L^\infty(\Omega)$, let $\phi(x,y)$ satisfy (3.57), and let K_Ω be the Biot-Savart kernel given by (3.58). Then for all time $t \geq 0$, we have*

$$u(t,x) = K_\Omega[\omega] = \int_\Omega K_\Omega(x,y)\omega(t,y)\,dy.$$

We note that if $\Omega = B(0,1)$, then $R(x,y) = K(x-y^*)$, as in Theorem 2.5.2.

Since Ω is simply connected, we could use a Riemann map to obtain an expression for $R(x,y)$, but we need only that this term has enough regularity to allow the above proof of Theorem 1.6.1 to carry through. The function $\phi(x,y)$ is C^∞ on $\Omega \times \Omega$ and $R(x,y)$ is smooth and bounded for y in the compact support of $\omega(t,y)$. Specifically, we have

$$\sup_{x\in\Omega, y\in \operatorname{supp}\omega(t,\cdot)} R(x,y) \le C_T,$$

$$\sup_{x\in\Omega, y\in \operatorname{supp}\omega(t,\cdot)} \nabla_x R(x,y) \le C_T.$$

Since these were the only properties of $K(x-y^*)$ used above, the proof of Theorem 1.6.1 for the arbitrary domain Ω is complete.

3.7 Serfati's Lemma

We will close this chapter by examining Serfati's Lemma in more detail and describing how it is used to obtain the crucial bound (3.47). The version of the lemma presented here is proven in [BK15] for $B \in M_{d\times d}(\mathbb{R})$, the space of d-dimensional real square matrices, for $d \ge 1$; we present it with $d = 2$. The result was further refined as Lemma 5.1 of [BK21], where the authors attribute the lemma to Serfati in [Ser94b, Ser92, Ser94a].

Lemma 3.7.1 (Serfati's Lemma) *For any symmetric matrix $B \in M_{2\times 2}(\mathbb{R})$, we have*

$$|B| \le \frac{P(M_1)}{|M_1|^4} |BM_1| + 2\,|\operatorname{tr} B|\,,$$

where M_1 is any vector in $\mathbb{R}^2$ and P is a polynomial of degree 4.

While we do not reproduce it here, we note that the proof of Lemma 3.7.1 is essentially a direct calculation applied to a representation of B as a product involving the matrix $M := (M_1, M_1^\perp)$, where $M_1^\perp := (-M_1^2, M_1^1)$. Because $MM^T = (\det M)I$, B can be represented as

$$B = \frac{MM^T}{\det M} B \frac{MM^T}{\det M} = \frac{M}{(\det M)^2}\left(M^T B M\right) M^T.$$

The proof is completed by using the symmetry of B to directly estimate $\left|M^T B M\right|$ in terms of $|BM_1|$ and $\operatorname{tr} B$.

In the proof of Theorem 1.6.1, we first obtained a rough estimate on the velocity gradient with (2.21) and (3.39) as

$$\|\nabla u_n(t)\|_{L^\infty} \leq V_n(t) := \|\omega_n(t)\|_{L^\infty} + \left\|\text{p. v.} \int_\Omega \nabla_x K_\Omega(x,y)\omega_n(t,y)\,dy\right\|_{L^\infty}.$$

In Section 3.5, we wanted to improve this estimate to bound $V_n(t)$ in terms of $\|Y_n(t)\|_{C^\alpha(\Omega)}$, which was itself bounded in terms of V_n in Section 3.4. This allowed us to close all the estimates with Grönwall's inequality in Section 3.6 in order to complete the proof of the main result. To this end, the principle value integral appearing in the expression for V_n was decomposed into three separate integrals, two of which were relatively straightforward to estimate. The third integral was the most difficult to handle because it dealt with the singularity at origin. This is where Lemma 3.7.1 was used.

The quantity to be estimated was

$$\left|\text{p. v.} \int_{\mathbb{R}^2} \nabla(a_r K)(x-y)\widetilde{\omega}_n(t,y)\,dy\right|.$$

As noted above, because this integral is over the whole plane and because we are using the extensions $\widetilde{\omega}_n$ and $\widetilde{Y}_n$ defined on the whole plane, the calculations in Section 10.3 of [BK15] apply directly. We will trace through their arguments in order to illustrate how Serfati's lemma gave the necessary bound.

To deal with the singularity at $x = y$, we use the cutoff functions a_r given by (2.7) to construct the following smooth radially symmetric double cutoff functions for $r, h > 0$

with $2h < r$:

$$\mu_{rh} = a_r(1 - a_h). \tag{3.59}$$

Radially, μ_{rh} is identically zero on the intervals $[0, h]$ and $[2r, \infty)$, is identically 1 on $[2h, r]$, and is smooth on $[h, 2h]$ and $[r, 2r]$. It follows ([GT01, Section 8.3]) that the gradient of μ_{rh} satisfies the following:

$$|\nabla \mu_{rh}| \leq \frac{C}{h} \leq \frac{C}{|x|} \quad \text{for } |x| \in (h, 2h) \tag{3.60}$$

$$|\nabla \mu_{rh}| \leq \frac{C}{r} \leq \frac{C}{|x|} \quad \text{for } |x| \in (r, 2r) \tag{3.61}$$

$$\nabla \mu_{rh} = 0 \quad \text{elsewhere} \tag{3.62}$$

The quantity to be estimated can now be written ([BK15, Proposition 4.7]) as

$$\left| \text{p. v.} \int_{\mathbb{R}^2} \nabla(a_r K)(x - y) \widetilde{\omega}_n(t, y)\, dy \right| = \left| \lim_{h \to 0} \nabla \left(\mu_{rh} K \right) * \widetilde{\omega}_n \right|.$$

Now let $\mathcal{F}(x) = \frac{1}{2\pi} \ln |x|$ be the fundamental solution to the Laplacian in $\mathbb{R}^2$. Recall that $K = \nabla^\perp \mathcal{F}$. Since, up to absolute value, it only reorders the components, we can replace K with $\nabla \mathcal{F}$ to see that the quantity to be estimated can be written as

$$\left| \text{p. v.} \int_{\mathbb{R}^2} \nabla(a_r K)(x - y) \widetilde{\omega}_n(t, y)\, dy \right| = \left| \lim_{h \to 0} \nabla \left[\mu_{rh} \nabla \mathcal{F} \right] * \widetilde{\omega}_n \right| = \lim_{h \to 0} |B|,$$

where

$$B = B(t, x) := \nabla \left[\mu_{rh} \nabla \mathcal{F} \right] * \widetilde{\omega}_n = \begin{pmatrix} \partial_1 \left[\mu_{rh} \partial_1 \mathcal{F} \right] * \widetilde{\omega}_n & \partial_2 \left[\mu_{rh} \partial_1 \mathcal{F} \right] * \widetilde{\omega}_n \\ \partial_1 \left[\mu_{rh} \partial_2 \mathcal{F} \right] * \widetilde{\omega}_n & \partial_2 \left[\mu_{rh} \partial_2 \mathcal{F} \right] * \widetilde{\omega}_n \end{pmatrix}.$$

Because $\nabla \nabla \mathcal{F}$ is not uniformly integrable as $h \to 0$, we cannot directly estimate $|B|$. However, we can apply Lemma 3.7.1 using $M_1 = \widetilde{Y}_n(t, x)$, defined on the whole plane by

(3.45), to obtain the bound

$$|B| \leq C\frac{P(\widetilde{Y}_n)}{\left|\widetilde{Y}_n\right|^4}\left|B\widetilde{Y}_n\right| + 2\left|\operatorname{tr} B\right|. \tag{3.63}$$

Since the trace of B does not depend on $\widetilde{Y}_n$, it can be estimated directly by writing it as

$$\begin{aligned}
\operatorname{tr} B &= \partial_1\left[\mu_{rh}\partial_1\mathcal{F}\right] * \widetilde{\omega}_n + \partial_2\left[\mu_{rh}\partial_2\mathcal{F}\right] * \widetilde{\omega}_n \\
&= \left[\partial_1\mu_{rh}\partial_1\mathcal{F} + \partial_2\mu_{rh}\partial_2\mathcal{F}\right] * \widetilde{\omega}_n + \left[\mu_{rh}\Delta\mathcal{F}\right] * \widetilde{\omega}_n \\
&= \left[\partial_1\mu_{rh}\partial_1\mathcal{F} + \partial_2\mu_{rh}\partial_2\mathcal{F}\right] * \widetilde{\omega}_n + \left[\mu_{rh}\delta_0\right] * \widetilde{\omega}_n,
\end{aligned}$$

where δ_0 is Dirac's delta function and $\Delta\mathcal{F} = \delta_0$ by definition. Since $\mu_{rh}(0) = 0$, we have $[\mu_{rh}\delta_0] * \widetilde{\omega}_n = 0$ so that

$$\operatorname{tr} B = \left[\partial_1\mu_{rh}\partial_1\mathcal{F} + \partial_2\mu_{rh}\partial_2\mathcal{F}\right] * \widetilde{\omega}_n.$$

Switching to polar coordinates and using properties (3.60) through (3.62), Property 3 of Proposition 3.2.1, and (3.1), we can now estimate the components of $\operatorname{tr} B$ for $j = 1, 2$ as

$$\begin{aligned}
\left|\left[\partial_j\mu_{rh}\partial_j\mathcal{F}\right] * \widetilde{\omega}_n\right| &= \left|\int_{\mathbb{R}^2}\left[\partial_j\mu_{rh}\partial_j\mathcal{F}\right](x-y)\widetilde{\omega}_n(t,y)\,dy\right| \\
&\leq C\left\|\widetilde{\omega}_n(t)\right\|_{L^\infty}\left[\left(\int_h^{2h} + \int_r^{2r}\right)\left|\left(\partial_j\mu_{rh}\right)(\rho)\right|\left|\left(\partial_j\mathcal{F}\right)(\rho)\right|\,\rho d\rho\right] \\
&\leq C\left\|\widetilde{\omega}_0\right\|_{L^\infty}\left[\frac{1}{h}\int_h^{2h}\left|\left(\partial_j\mathcal{F}\right)(\rho)\right|\,\rho d\rho + \frac{1}{r}\int_r^{2r}\left|\left(\partial_j\mathcal{F}\right)(\rho)\right|\,\rho d\rho\right] \\
&\leq C\left\|\widetilde{\omega}_0\right\|_{L^\infty}\left[\frac{1}{h}\int_h^{2h}\frac{1}{\rho}\,\rho d\rho + \frac{1}{r}\int_r^{2r}\frac{1}{\rho}\,\rho d\rho\right] \\
&\leq C\left\|\widetilde{\omega}_0\right\|_{L^\infty}
\end{aligned}$$

uniformly over $h < \frac{r}{2}$. This gives that

$$\lim_{h\to 0}\left|\operatorname{tr} B\right| \leq C\left\|\widetilde{\omega}_0\right\|_{L^\infty}. \tag{3.64}$$

It remains to estimate $\left|B\widetilde{Y}_n\right|$. This matrix is given by

$$B\widetilde{Y}_n = \begin{pmatrix} R_1 \\ \\ R_2 \end{pmatrix} = \begin{pmatrix} \left(\partial_1 \left[\mu_{rh}\partial_1\mathcal{F}\right] * \widetilde{\omega}_n\right)\widetilde{Y}_n^1 & \left(\partial_2 \left[\mu_{rh}\partial_1\mathcal{F}\right] * \widetilde{\omega}_n\right)\widetilde{Y}_n^2 \\ \\ \left(\partial_1 \left[\mu_{rh}\partial_2\mathcal{F}\right] * \widetilde{\omega}_n\right)\widetilde{Y}_n^1 & \left(\partial_2 \left[\mu_{rh}\partial_2\mathcal{F}\right] * \widetilde{\omega}_n\right)\widetilde{Y}_n^2 \end{pmatrix}.$$

We can add and subtract a quantity and rearrange the terms to further decompose each row R_j as

$$\begin{aligned} R_j &= \left(\partial_1 \left[\mu_{rh}\partial_j\mathcal{F}\right] * \widetilde{\omega}_n\right)\widetilde{Y}_n^1 + \left(\partial_2 \left[\mu_{rh}\partial_j\mathcal{F}\right] * \widetilde{\omega}_n\right)\widetilde{Y}_n^2 \\ &\quad - \partial_1 \left[\mu_{rh}\partial_j\mathcal{F}\right] * \left(\widetilde{\omega}_n\widetilde{Y}_n^1\right) - \partial_2 \left[\mu_{rh}\partial_j\mathcal{F}\right] * \left(\widetilde{\omega}_n\widetilde{Y}_n^2\right) \\ &\quad + \partial_1 \left[\mu_{rh}\partial_j\mathcal{F}\right] * \left(\widetilde{\omega}_n\widetilde{Y}_n^1\right) + \partial_2 \left[\mu_{rh}\partial_j\mathcal{F}\right] * \left(\widetilde{\omega}_n\widetilde{Y}_n^2\right) \\ &= p_j + q_j, \end{aligned} \tag{3.65}$$

where

$$\begin{aligned} p_j &:= \left(\partial_1 \left[\mu_{rh}\partial_j\mathcal{F}\right] * \widetilde{\omega}_n\right)\widetilde{Y}_n^1 + \left(\partial_2 \left[\mu_{rh}\partial_j\mathcal{F}\right] * \widetilde{\omega}_n\right)\widetilde{Y}_n^2 \\ &\quad - \partial_1 \left[\mu_{rh}\partial_j\mathcal{F}\right] * \left(\widetilde{\omega}_n\widetilde{Y}_n^1\right) - \partial_2 \left[\mu_{rh}\partial_j\mathcal{F}\right] * \left(\widetilde{\omega}_n\widetilde{Y}_n^2\right) \\ &= \sum_{k=1,2} \left[\left(\partial_k \left[\mu_{rh}\partial_j\mathcal{F}\right] * \widetilde{\omega}_n\right)\widetilde{Y}_n^k - \partial_k \left[\mu_{rh}\partial_j\mathcal{F}\right] * \left(\widetilde{\omega}_n\widetilde{Y}_n^k\right)\right] \end{aligned}$$

and

$$q_j := \partial_1 \left[\mu_{rh}\partial_j\mathcal{F}\right] * \left(\widetilde{\omega}_n\widetilde{Y}_n^1\right) + \partial_2 \left[\mu_{rh}\partial_j\mathcal{F}\right] * \left(\widetilde{\omega}_n\widetilde{Y}_n^2\right).$$

Thus, in order to estimate $\left|B\widetilde{Y}_n\right|$, it is sufficient to estimate $|p_j|$ and $|q_j|$.

Recall that an immediate consequence of Definition 2.1.1 is that, for any $f \in C^\alpha$, we have

$$|f(x) - f(y)| \leq \|f\|_{C^\alpha} |x-y|^\alpha. \tag{3.66}$$

Using (3.66), the support of μ_{rh}, and polar coordinates, we can estimate

$$\begin{aligned}|p_j| &= \left|\sum_{k=1,2}\int_{\mathbb{R}^2}\left(\partial_k\left[\mu_{rh}\partial_j\mathcal{F}\right]\right)(x-y)\widetilde{\omega}_n(t,y)\left[\widetilde{Y}_n^k(t,x)-\widetilde{Y}_n^k(t,y)\right]\,dy\right|\\
&\le 2\int_{\mathbb{R}^2}\left|\left(\nabla\left[\mu_{rh}\nabla\mathcal{F}\right]\right)(x-y)\right|\left|\widetilde{\omega}_n(t,y)\right|\left|\widetilde{Y}_n^k(t,x)-\widetilde{Y}_n^k(t,y)\right|\,dy\\
&\le C\left\|\widetilde{Y}_n(t)\right\|_{C^\alpha}\left\|\widetilde{\omega}_n(t)\right\|_{L^\infty}\int_h^{2r}\left|\left(\nabla\left[\mu_{rh}\nabla\mathcal{F}\right]\right)(\rho)\right|\rho^\alpha\,\rho d\rho.\end{aligned}$$

Since $\nabla\left[\mu_{rh}\nabla\mathcal{F}\right]=\mu_{rh}\nabla\nabla\mathcal{F}+\nabla\mu_{rh}\nabla\mathcal{F}$, we have that

$$\left|\nabla\left[\mu_{rh}\nabla\mathcal{F}\right](\rho)\right|\le\frac{C}{\rho^2}. \tag{3.67}$$

This gives

$$\begin{aligned}|p_j| &\le C\left\|\widetilde{Y}_n(t)\right\|_{C^\alpha}\left\|\widetilde{\omega}_0\right\|_{L^\infty}\int_h^{2r}\frac{1}{\rho^2}\rho^\alpha\,\rho d\rho\\
&= C(\alpha)\left\|\widetilde{Y}_n(t)\right\|_{C^\alpha}\left\|\widetilde{\omega}_0\right\|_{L^\infty}\left[r^\alpha-h^\alpha\right]\end{aligned}$$

so that

$$\sum_{j=1,2}\left|\lim_{h\to0}p_j\right|\le C\left\|\widetilde{Y}_n(t)\right\|_{C^\alpha}\left\|\widetilde{\omega}_0\right\|_{L^\infty}r^\alpha. \tag{3.68}$$

We now turn to $|q_j|$, our last remaining term to estimate. First, we can use properties of convolution to write

$$\begin{aligned}q_j &= \partial_1\left[\mu_{rh}\partial_j\mathcal{F}\right]*\left(\widetilde{\omega}_n\widetilde{Y}_n^1\right)+\partial_2\left[\mu_{rh}\partial_j\mathcal{F}\right]*\left(\widetilde{\omega}_n\widetilde{Y}_n^2\right)\\
&= \left(\mu_{rh}\partial_j\mathcal{F}\right)*\partial_1\left(\widetilde{\omega}_n\widetilde{Y}_n^1\right)+\left(\mu_{rh}\partial_j\mathcal{F}\right)*\partial_2\left(\widetilde{\omega}_n\widetilde{Y}_n^2\right)\\
&= \left(\mu_{rh}\partial_j\mathcal{F}\right)*\operatorname{div}\left(\widetilde{\omega}_n\widetilde{Y}_n\right),\end{aligned}$$

giving that

$$\sum_{j=1,2}|q_j|\le C\left|\lim_{h\to0}\int_{\mathbb{R}^2}\left(\mu_{rh}\nabla\mathcal{F}\right)(x-y)\operatorname{div}\left(\widetilde{\omega}_n\widetilde{Y}_n\right)(y)\,dy\right|.$$

By Proposition 3.3.2, $\operatorname{div}\left(\widetilde{\omega}_n \widetilde{Y}_n\right) \in C^{\alpha-1}$ so we can write it as $\operatorname{div}\left(\widetilde{\omega}_n \widetilde{Y}_n\right) = f_0 + \operatorname{div} f_1$, where $f_0, f_1 \in C^\alpha$. A direct consequence of Definition 2.1.1 is that

$$\|f_0\|_{C^\alpha}, \|f_1\|_{C^\alpha} \le C \left\|\operatorname{div}\left(\widetilde{\omega}_n \widetilde{Y}_n\right)\right\|_{C^{\alpha-1}}. \tag{3.69}$$

This allows us to decompose the quantity to be estimated as

$$\begin{aligned}\sum_{j=1,2} |q_j| &\le C \left|\int_{\mathbb{R}^2} (\mu_{rh} \nabla \mathcal{F})(x-y) f_0(y)\, dy\right| + C \left|\int_{\mathbb{R}^2} (\mu_{rh} \nabla \mathcal{F})(x-y) (\operatorname{div} f_1)(y)\, dy\right| \\ &:= I + II. \end{aligned} \tag{3.70}$$

A basic property of $\nabla \mathcal{F}$ is that its mean value over any circle is zero, so that $\int_{\mathbb{R}^2} (\mu_{rh} \nabla \mathcal{F})(x-y) f_0(x)\, dy = 0$. This allows us to estimate I as

$$\begin{aligned}|I| &= C \left|\int_{\mathbb{R}^2} (\mu_{rh} \nabla \mathcal{F})(x-y) (f_0(y) - f_0(x))\, dy\right| \\ &\le C \|f_0\|_{C^\alpha} \int_h^{2r} \rho^{-1} \rho^\alpha \rho d\rho \\ &\le C(\alpha) \|f_0\|_{C^\alpha} r^{\alpha+1} \\ &\le C \left\|\operatorname{div}\left(\widetilde{\omega}_n \widetilde{Y}_n\right)\right\|_{C^{\alpha-1}} r^\alpha, \end{aligned} \tag{3.71}$$

where we used (3.66), the support of μ_{rh}, (3.1), (3.69), and the fact that $r \le 1$ (which follows from the ultimate choice of r made at the beginning of Section 3.6).

We can estimate II in a similar manner as

$$\begin{aligned}|II| &= C \left|\int_{\mathbb{R}^2} (\mu_{rh} \nabla \mathcal{F})(x-y) (\operatorname{div} f_1)(y)\, dy\right| \\ &= C \left|\int_{\mathbb{R}^2} (\mu_{rh} \nabla \mathcal{F})(x-y) \operatorname{div}_y (f_1(y) - f_1(x))\, dy\right| \\ &\le C \left|\int_{\mathbb{R}^2} \nabla [\mu_{rh} \nabla \mathcal{F}](x-y) (f_1(y) - f_1(x))\, dy\right| \end{aligned}$$

$$\leq C \left\| f_1 \right\|_{C^\alpha} \int_h^{2r} \rho^{-2} \rho^\alpha \, \rho d\rho$$

$$\leq C \left\| f_1 \right\|_{C^\alpha} r^\alpha$$

$$\leq C \left\| \operatorname{div} \left(\widetilde{\omega}_n \widetilde{Y}_n \right) \right\|_{C^{\alpha-1}} r^\alpha, \tag{3.72}$$

where integrated by parts and used (3.66), the support of μ_{rh}, (3.67), and (3.69).

Putting (3.71) and (3.72) together with (3.70) gives

$$\sum_{j=1,2} \left| \lim_{h \to 0} q_j \right| \leq C \left\| \operatorname{div} \left(\widetilde{\omega}_n \widetilde{Y}_n \right) \right\|_{C^{\alpha-1}} r^\alpha. \tag{3.73}$$

Using (3.68) and (3.73) together with (3.65) gives us the estimate

$$\left| B \widetilde{Y}_n \right| \leq C \left\| \widetilde{Y}_n(t) \right\|_{C^\alpha} \left\| \widetilde{\omega}_0 \right\|_{L^\infty} r^\alpha + C \left\| \operatorname{div} \left(\widetilde{\omega}_n \widetilde{Y}_n \right) (t) \right\|_{C^{\alpha-1}} r^\alpha. \tag{3.74}$$

Using (3.64) and (3.74) with (3.63) gives the estimate (appearing as (10.16) in [BK15])

$$\lim_{h \to 0} |B| \leq C \frac{P(Y_n)}{|Y_n|^4} \left(\left\| \widetilde{Y}_n \right\|_{C^\alpha} \left\| \widetilde{\omega}_0 \right\|_{L^\infty} + \left\| \operatorname{div} \left(\widetilde{\omega}_n \widetilde{Y}_n \right) \right\|_{C^{\alpha-1}} \right) r^\alpha + C \left\| \widetilde{\omega}_0 \right\|_{L^\infty}.$$

Applying (3.44) and recalling that Lemma 3.7.1 specified that P is of degree 4 yields the estimate given as (10.17) in [BK15] and as (3.46) in Section 3.5 above, from which point the proof of Theorem 1.6.1 continued.

Chapter 4

Conclusions

In this chapter, we will present some of the applications for which Theorem 1.6.1 provides a solution. We will also discuss the limitations of Theorem 1.6.1 and some possible avenues of future investigation.

4.1 Applications of Theorem 1.6.1

Classical Vortex Patches

A classical vortex patch occurs when the initial velocity $\omega(0,x) = b\mathbb{1}_U$, where b is a constant and U is a simply connected domain. The following theorem reproduces the bounded domain vortex patch regularity result from [Dep98] using Theorem 1.6.1.

Corollary 4.1.1 (Propagation of Boundary Regularity for Vortex Patches in Ω) *Let Ω be an simply connected bounded domain in $\mathbb{R}^2$ with a C^∞ boundary. Let $U \subseteq \Omega$ be a simply connected domain such that* $\operatorname{dist}(\partial\Omega, U) = \delta > 0$ *and let $\omega_0(x) = \omega(0,x) = b\mathbb{1}_U$. If ∂U is $C^{1,\alpha}$, then the boundary of $U_t := \eta(t, \partial U)$ remains $C^{1,\alpha}$ for all time.*

Proof. Since $\partial U \in C^{1,\alpha}$, we can take a scalar $\varphi_0 \in C^{1,\alpha}(\Omega)$ with the following properties:

$$\begin{aligned}
&\varphi_0(x) > 0 \quad \text{in } U, \\
&\varphi_0(x) = 0 \quad \text{on } \partial U, \\
&\inf_{x\in\partial U} |\nabla\varphi_0(x)| \geq 2\gamma > 0.
\end{aligned} \tag{4.1}$$

Let $Y_0 = \nabla^{\perp}\varphi_0$. Since $\varphi_0 \in C^{1,\alpha}(\Omega)$, we have $Y_0 \in C^{\alpha}(\Omega)$. Note that Y_0 is tangential to the boundary of the vortex patch U and that $\operatorname{div} Y_0 = \operatorname{div} \nabla^{\perp}\varphi_0 = 0$, so Y_0 is divergence-free (and hence, $\operatorname{div} Y_0 \in C^{\alpha}(\Omega)$). Since the boundary of the patch U is compact in Ω, we have that $Y_0 \geq \gamma > 0$ on some δ_0-neighborhood $\mathcal{N}_{\delta_0}(\partial U) := \{x \in \Omega : \operatorname{dist}(x, \partial U) < \delta_0\}$ of ∂U, where $\delta_0 < \delta/2$. However, by construction, Y_0 necessarily must vanish for at least one point inside Ω. Because of this, we include an auxiliary vector field Y^* on Ω to form our sufficient family. We take Y^* to be an arbitrary non-vanishing divergence-free smooth vector field on Ω; for example, we could use $Y^*(x_1, x_2) = (1, x_1)$. We note that any conclusions drawn from Theorem 1.6.1 about Y^* are inconsequential and that its only purpose is to "fill out" our sufficient family to avoid the issue with the vanishing gradient of φ_0. Since Y^* is C^{∞}, we have that $Y^*, \operatorname{div} Y^* \in C^{\alpha}(\Omega)$. Since we now have ensured that $I_{\Omega}\{Y_0, Y^*\} \geq c > 0$, we have verified that $\mathcal{Y}_0 := \{Y_0, Y^*\}$ is a C^{α} sufficient family on Ω.

We now consider the hypotheses of Theorem 1.6.1. We clearly have $\omega_0 \in L^{\infty}(\Omega)$ and ω_0 compactly supported. Formally, we have $\operatorname{div}(\omega_0 Y_0) = \omega_0 \operatorname{div} Y_0 + \nabla\omega_0 \cdot Y_0 = 0$. More precisely, let $\phi \in C_c^{\infty}(\Omega)$ be a test function. Then

$$\begin{aligned}
\int_{\Omega} \operatorname{div}(\omega_0 Y_0)(x)\phi(x)\,dx &= -\int_{\Omega} \omega_0(x) Y_0(x) \cdot \nabla\phi(x)\,dx \\
&= -b\int_U Y_0(x) \cdot \nabla\phi(x)\,dx
\end{aligned}$$

$$= b \int_U \operatorname{div} Y_0(x)\phi(x)\,dx$$

$$= 0.$$

This shows that $\operatorname{div}(\omega_0 Y_0) = 0 \in C^{\alpha-1}$. An identical argument applied to Y^* shows that $\operatorname{div}(\omega_0 \mathcal{Y}_0) \in C^{\alpha-1}$. By Theorem 1.3 of [BK15], this is equivalent to having $\mathcal{Y}_0 \cdot \nabla u \in C^\alpha$, so the hypotheses of Theorem 1.6.1 are satisfied.

We now follow the strategy of [BC93] and let $\varphi(t)$ be φ_0 transported by the flow map so that

$$\partial_t \varphi + u \cdot \nabla \varphi = 0, \tag{4.2}$$

implying that $\varphi(t,x) = \varphi_0(\eta^{-1}(t,x))$. By taking the perpendicular gradient of (4.2), we can compute (for example, this is equation 3.4 of [BC93]) that

$$\partial_t \nabla^\perp \varphi + u \cdot \nabla \nabla^\perp \varphi = \nabla^\perp \varphi \cdot \nabla u.$$

Comparing this to (3.12) and (3.14), we see that $Y_0(t) := \nabla^\perp \varphi(t)$ is the pushforward of Y_0 to time t by the flow map. By Theorem 1.6.1, we have that $Y_0(t) \in C^\alpha(\Omega)$ for all time, so that $\varphi(t) \in C^{1,\alpha}$ for all time. Because ∂U_t remains a level set of $\varphi(t)$, we have that $Y(t) = \nabla^\perp \varphi(t)$ remains tangential to ∂U_t. Therefore, the boundary of the vortex patch remains $C^{1,\alpha}$ for all time, proving the persistence of regularity of vortex patch boundaries in a bounded domain. ■

Sum of Disjoint Vortex Patches

Now suppose that we have a finite number of pairwise disjoint open simply connected sets $U_1, \dots, U_k$ in Ω, all with $C^{1,\alpha}$ boundaries, satisfying

$$D_\Omega := \min_i \operatorname{dist}(U_i, \partial\Omega) \geq \delta_\Omega > 0,$$

and

$$D := \min_{i \neq j} \operatorname{dist}(U_i, U_j) \geq \delta > 0.$$

Here, D_Ω is the distance from the union of the sets U_i to the boundary of the domain and D is the smallest distance between any two of the open sets. Let $U_{i,t} = \eta(t, U_i)$ be the image of U_i under the flow map after time t.

Consider an initial vorticity given by a sum of vortex patches over each open set, and write it as

$$\omega_0(x) = \sum_{i=1}^{k} b_i \mathbb{1}_{U_i},$$

where each b_i is a constant. Because $\partial U_i \in C^{1,\alpha}$, for each U_i, we can choose a scalar $\varphi_{i,0} \in C^{1,\alpha}(\Omega)$ with the following properties:

$$\varphi_{i,0}(x) > 0 \quad \text{in } U_i,$$

$$\varphi_{i,0}(x) = 0 \quad \text{on } \partial U_i,$$

$$\inf_{x \in \partial U_i} |\nabla \varphi_{i,0}(x)| \geq 2\gamma > 0.$$

Using smooth cutoff functions if necessary, we can also choose each $\varphi_{i,0}$ to be such that $\operatorname{supp} \varphi_{i,0} \subseteq \mathcal{N}_D(U_i)$, so that the functions $\varphi_{i,0}$ have pairwise disjoint supports.

We can then define the vector fields $Y_{i,0} := \nabla^{\perp}\varphi_{i,0}$. As before, for each $i = 1, \dots, k$, we have $Y_{i,0} \in C^{\alpha}(\Omega)$ and $\operatorname{div} Y_{i,0} = 0 \in C^{\alpha}$. We again take an arbitrary non-vanishing divergence-free smooth vector field Y^* to fill out our sufficient family, and define $\mathcal{Y}_0 = \{Y_{1,0}, \dots, Y_{k,0}, Y^*\}$. Thus, $\mathcal{Y}_0$ is a C^{α} sufficient family on Ω.

We clearly have $\omega_0 \in L^{\infty}(\Omega)$ and is compactly supported in the closure of the union of the U_i. The calculations verifying that $\mathcal{Y}_0 \cdot \nabla u \in C^{\alpha}$ carry out identically to the previous case, showing that the hypotheses of Theorem 1.6.1 are satisfied. By letting each $\varphi_{i,0}$ be transported by the flow to obtain $\varphi_i(t)$ satisfying

$$\partial_t \varphi_i + u \cdot \nabla \varphi_i = 0,$$

we can take the perpendicular gradient as before to find that $Y_i(t) = \nabla^{\perp}\varphi_i(t)$. Thus, we can apply Theorem 1.6.1 to get that, for all i, $Y_i(t) \in C^{\alpha}(\Omega)$ for all time, so that $\varphi_i(t) \in C^{1,\alpha}$ for all time. As in the classical vortex patch case, we also have that $\nabla^{\perp}\varphi_i(t)$ remains tangential to the boundary of $\partial U_{i,t}$ for all time. Therefore, a finite sum of classical vortex patches with initial boundaries in $C^{1,\alpha}$ will maintain their boundary regularity for all time.

Patches of Non-Constant Vorticity

Let U be an open simply connected set in Ω with $\partial U \in C^{1,\alpha}$ and let $f \in C^{\alpha}(\Omega)$ with $f\big|_{\partial U} = b$, a constant. Consider the initial vorticity $\omega_0 = f \mathbb{1}_U$. Then we can think of ω_0 as a vortex patch of non-constant vorticity.

As with the classical vortex patch case, we can choose a scalar $\varphi_0 \in C^{1,\alpha}(\Omega)$ satisfying (4.1), set $Y_0 = \nabla^{\perp}\varphi_0$, and take an auxiliary smooth non-vanishing divergence-free vector field Y^*. We take $\mathcal{Y}_0 = \{Y_0, Y^*\}$ as our sufficient family.

Consider the function $\omega_0 - b\mathbb{1}_U$, which can be thought of as "translating the height" of the vortex patch so that it is equal to $f - b$ in U and is continuous on the boundary of U. This gives that $\omega_0 - b\mathbb{1}_U \in C^\alpha(\Omega)$, as is Y_0, so that $\operatorname{div}((\omega_0 - b\mathbb{1}_U)Y_0) \in C^{\alpha-1}$. However,

$$\operatorname{div}((\omega_0 - b\mathbb{1}_U)Y_0) = \operatorname{div}(\omega_0 Y_0) - b \operatorname{div}(\mathbb{1}_U Y_0) = \operatorname{div}(\omega_0 Y_0)$$

since in the classical vortex patch case we showed that $\operatorname{div}(\mathbb{1}_U Y_0) = 0$, so that $\operatorname{div}(\omega_0 \mathcal{Y}_0) \in C^{\alpha-1}$. Thus, Theorem 1.6.1 can be applied to see that $Y(t) = \nabla^\perp \varphi_t \in C^\alpha(\Omega)$ for all time, so that the boundary of $U_t = \eta(t, U)$ will remain $C^{1,\alpha}$ for all time. Thus, we have shown that patches of non-constant vorticity maintain their $C^{1,\alpha}$ boundary regularity for all time. The same modifications used in the case of a sum of disjoint vortex patches could also be applied to show that a finite sum of disjoint patches of non-constant vorticity maintain $C^{1,\alpha}$ boundary regularity for all time.

4.2 Future Work

One of the hypotheses of Theorem 1.6.1 is that the initial vorticity be supported away from the boundary of the domain. It remains to be seen whether this is strictly necessary for global regularity of vortex patch boundaries in bounded domains. This assumption ensures that the Biot-Savart integral in Theorem 2.5.2 exists since $K_\Omega(x, y)$ is singular along the boundary of Ω. Further, the compact support of ω_0 was explicitly used in the regularization scheme presented in Section 3.2, where the compact support allowed us to extend ω_0 by zero to the whole plane without changing any of its norms and to use the mollifiers ρ_n. If ω_0 was allowed to touch the boundary, even at a single point, while it may be possible to argue that the Biot-Savart integral would still be convergent, it would be difficult to smooth the initial data in a simple way since mollification would extend the support of the smoothed data outside of the domain. Since the majority of the proof is carried out with the smooth approximations, this presents a significant problem that would need to be solved. Despite these issues, in [Dep99], Depauw was able to prove *local-in-time* persistence of vortex patch boundary regularity when the initial patch is tangent to the boundary using the paradifferential calculus methods of Chemin.

Another avenue of future exploration would be the expansion of the results to a higher-dimensional setting. In higher dimensions, the vorticity is no longer a scalar, but is vector-valued instead, so simple vortex patches in the sense of indicator functions of simply connected bounded domains do not exist. Indeed, as the discussion immediately preceding Definition 1.1 of [GSR95] shows, it is not possible to have a constant vorticity field supported on a bounded domain in three or more dimensions. However, it may be possible to prove striated regularity in a bounded domain in higher dimensions. In [BK15], they prove whole-space analogues of Theorem 1.6.1 for $\mathbb{R}^d$, with $d \geq 2$, as Theorem 1.1. Due to higher dimensional phenomena such as vortex stretching, the $\mathbb{R}^d$ setting is more difficult to handle. One fact unique to the two-dimensional case that was used extensively in this work is that the vorticity is conserved along particle trajectories since $\partial_t \omega + u \cdot \nabla \omega = 0$. This allowed us to trace vorticity estimates at time t back to estimates of ω_0. This is not the case in higher dimensions, where we have $\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u$ (see, for example, Proposition 1.8 of [MB02]). As a result, more technical approaches would be required to investigate the problem in higher dimensions. In addition, the relatively simple properties of the Biot-Savart corrector term $R(x, y)$ in two dimensions from (3.58) allowed us to adapt the $\mathbb{R}^2$

results and calculations from [BK15] to our bounded domain setting mostly by dealing with sometimes difficult, but manageable, extra integral terms coming from the Biot-Savart Law, Theorem 2.5.2. Again, this situation is more complex in higher dimensions, so it is unclear whether the approach taken in this work would even be possible.

In the vortex patch applications in Section 4.1, one may notice that the interior of the vortex patch does not play a significant role. In the classical vortex patch case, the vorticity is constant in the interior, but we also showed that we can take it to be any C^α function inside the patch as long as it is constant along the boundary. This leads to the natural question of whether any vortex singularities could be allowed inside the boundary. Such a vorticity would not be α-Hölder continuous, so the arguments used in this work would not be sufficient, though it may be the case that they could be adapted to achieve this result.

Appendix A

Classical ODE Existence Theory and Proof of Proposition 3.3.2

We now summarize a few existence and uniqueness results from classical ODE theory used in this work (see [Eva10, MB02, Osg98]). Let $U \subseteq \mathbb{R}^d$ be a simply connected bounded open set with a C^∞ boundary. Let $u : [0, T] \times U \to \mathbb{R}^d$ satisfy $u \cdot \hat{n} = 0$ on the boundary ∂U. Let $z_0 \in U$. Consider the ordinary initial value problem

$$\begin{cases} z'(t) = u\left(t, z(t)\right), \\ z(t_0) = z_0. \end{cases} \tag{A.1}$$

Theorem A.1 (Cauchy-Lipschitz-Picard-Lindelöf Existence and Uniqueness) *If u is Lipschitz continuous in space, uniformly in time, then, for any time $T > 0$, (A.1) has a unique solution $z : [0, T] \to U$.*

Definition A.2 (Osgood's Condition) *A continuous non-decreasing modulus of continuity $\mu : [0, \infty) \to [0, \infty)$ with $\mu(0) = 0$ and $\mu(s) > 0$ when $s > 0$ is an Osgood modulus of* *continuity (or satisfies Osgood's condition) if*

$$\int_0^1 \frac{ds}{\mu(s)} = \infty.$$

Theorem A.3 *If u admits an Osgood modulus of continuity uniformly in time, then, for any time $T > 0$, (A.1) has a unique solution $z : [0,T] \to U$.*

Definition A.4 *A function $f : [0,T] \times U \to U$ is Log-Lipschitz (in space) if it admits the spatial modulus of continuity*

$$\mu_{LL}(r) = \begin{cases} -r \ln r & \text{if } r \leq e^{-1}, \\ e^{-1} & \text{if } r > e^{-1}, \end{cases}$$

bounded uniformly over $[0,T]$.

Remark A.5 *The modulus of continuity μ_{LL} satisfies Osgood's condition, so (A.1) has a unique solution for all time when u is Log-Lipschitz.*

Lemma A.6 (Reverse Osgood's Lemma) *Suppose $L : [0,\infty) \to (0,\infty)$ is differentiable and $\mu : (0,\infty) \to (0,\infty)$ is integrable with $L' \geq -\mu \circ L$. Then, for all $t > 0$,*

$$\int_{L(t)}^{L(0)} \frac{dr}{\mu(r)} \leq t.$$

Proof. Using a substitution, we can directly calculate that

$$-\int_{L(t)}^{L(0)} \frac{dr}{\mu(r)} = \int_{L(0)}^{L(t)} \frac{dr}{\mu(r)} = \int_0^t \frac{L'(s)}{\mu(L(s))}\, ds \geq \int_0^t \frac{-\mu(L(s))}{\mu(L(s))}\, ds = -t,$$

from which the lemma follows. ■

We now present the details of how to adapt Lemma 9.2 from [BK15] to the bounded domain setting to obtain the bound stated above as Proposition 3.3.2. To do so, the vector field Y_n must be extended from Ω to $\mathbb{R}^2$ in a way that gives us control over the divergence of the extension so that we may apply the result from [BK15], which the Stein Hölder extensions $\mathcal{E}_\alpha$ do not provide. Instead, we use a smooth cutoff function, which involves changing the value of Y_n near the boundary inside the domain Ω but outside the support of ω_n. While this allows us to obtain the necessary $C^{\alpha-1}$ bound on $\operatorname{div} \omega_n Y_n$, this extension is not used anywhere else in this work so the proof is provided here.

Proposition A.7 *We have* $\operatorname{div}(\omega_n Y_n)(t,\cdot) \in C^{\alpha-1}(\Omega)$ *with*

$$\|\operatorname{div}(\omega_n Y_n)(t,\cdot)\|_{C^{\alpha-1}(\Omega)} \leq Ce^{C\int_0^t \|\nabla u_n(s,\cdot)\|_{L^\infty(\Omega)}\,ds}.$$

Proof.

Recall that the smooth velocities $u_n(t,x) \in C^\infty(\Omega)$, so as outlined in Section 2.5, there exist stream functions $\psi_n(t,x) \in C^\infty(\Omega)$ given by (2.15) so that $u_n = \nabla^\perp \psi_n$. We now will use the Stein extension operator $\mathcal{E}$ for Ω and its properties given by Lemma 2.1.4. Recall that this operator simultaneously extends all functions in $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^2)$ for all k and p.

Define $\psi_n^* := \mathcal{E}\psi_n$. Since $\psi_n \in W^{2,\infty}(\Omega)$, we have that $\psi_n^* \in W^{2,\infty}(\mathbb{R}^2)$. We then define $u_n^* := \nabla^\perp \psi_n^*$ and $\omega_n^* = \operatorname{curl} u_n^*$, which gives that $u_n^* \in W^{1,\infty}(\mathbb{R}^2)$ and $\omega_n^* \in W^{0,\infty}(\mathbb{R}^2)$. By construction, we have $\operatorname{div} u_n^* = \operatorname{div} \nabla^\perp(\mathcal{E}\psi_n) = 0$, so that u_n^* is divergence-free. We note that, even though u_n^* and ω_n^* are not obtained by applying $\mathcal{E}$ to u_n and ω_n, respectively, we still have that $u_n^*\big|_\Omega = u_n$ and $\omega_n^*\big|_\Omega = \omega_n$, so that ω_n^* is compactly supported in Ω.

In order to extend Y_n to $\mathbb{R}^2$, we rely on the fact that this vector field is being multiplied by the compactly supported vorticity. Let $\delta = \operatorname{dist}(\operatorname{supp} \omega_n^*, \partial\Omega)$. We take a radially symmetric smooth cutoff function $\phi(r)$ that is equal to 1 on $B(0, 1-\delta/2)$, identically zero outside $B(0, 1-\delta/4)$, and decreasing for $1-\delta/2 < r < 1-\delta/4$. Note that ϕY_n is then compactly supported on Ω. We define the extension $\widetilde{Y_n} := \phi Y_n$, extended by zero outside Ω. By construction, we have $\widetilde{Y_n} \in C^\alpha$ and $\operatorname{div} \widetilde{Y_n} \in C^\alpha$. Note that this extension changes the value of Y_n inside Ω but not on the support of ω_n^*. Because of this, we are able to use it to obtain the necessary regularity result, but it is not suitable to be used elsewhere in the proof.

With these extensions in hand, we turn to the quantity $\operatorname{div}\left(\omega_n^* \widetilde{Y_n}\right)$. Because $\omega_n^* \in W^{0,\infty}(\mathbb{R}^2)$ and $\widetilde{Y_n} \in C^\alpha(\mathbb{R}^2)$, we have $\omega_n^* \widetilde{Y_n} \in C^\alpha(\mathbb{R}^2)$ as well. Let $\phi \in C_0^\infty(\mathbb{R}^2)$ be a compactly supported test function. Because $\widetilde{Y_n}$ is compactly supported in Ω and because ω_n^* and $\widetilde{Y_n}$ are extensions of ω_n and Y_n from $\operatorname{supp} \omega_n$ to $\mathbb{R}^2$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} \operatorname{div}\left(\omega_n^* \widetilde{Y_n}\right) \phi \, dx &= -\int_{\mathbb{R}^2} (\omega_n^* \widetilde{Y_n}) \cdot \nabla \phi \, dx \\
&= -\int_{\Omega} (\omega_n^* \widetilde{Y_n}) \cdot \nabla \phi \, dx \\
&= -\int_{\Omega} (\omega_n Y_n) \cdot \nabla \phi \, dx \\
&= \int_{\Omega} \operatorname{div}(\omega_n Y_n) \phi \, dx.
\end{aligned}
$$

Thus, $\operatorname{div}\left(\omega_n^* \widetilde{Y_n}\right)$ is well-defined as a distribution since $\operatorname{div}(\omega_n Y_n)$ is. This means that we can write $\operatorname{div}\left(\omega_n^* \widetilde{Y_n}\right)$ as $0 + \operatorname{div}\left(\omega_n^* \widetilde{Y_n}\right)$ where $\omega_n^* \widetilde{Y_n} \in C^\alpha(\mathbb{R}^2)$, so that $\operatorname{div}\left(\omega_n^* \widetilde{Y_n}\right) \in C^{\alpha-1}(\mathbb{R}^2)$, by definition (2.3). Further, the compact support of Y_n in Ω implies that $\omega_n Y_n$ and $\omega_n^* \widetilde{Y_n}$ are pointwise equal on their common support, which is contained in Ω. Recall that the definition of the $C^{\alpha-1}$ norm was given in (2.4) as

$$
\|h\|_{C^{\alpha-1}(U)} = \inf\{\|f\|_{C^\alpha(U)} + \|v\|_{C^\alpha(U)} : h = f + \operatorname{div} v; f, v \in C^\alpha(U)\}.
$$

Suppose that $\operatorname{div}\left(\omega_n^* \widetilde{Y_n}\right) = f + \operatorname{div} v$, where $f, v \in C^\alpha(\mathbb{R}^2)$. We have $\operatorname{div}(\omega_n Y_n) = \operatorname{div}\left(\left(\omega_n^* \widetilde{Y_n}\right)\Big|_\Omega\right) = f\big|_\Omega + \operatorname{div}\left(v\big|_\Omega\right)$. Since $\|f\|_{C^\alpha(\mathbb{R}^2)} \geq \left\|f\big|_\Omega\right\|_{C^\alpha(\Omega)}$ and $\|v\|_{C^\alpha(\mathbb{R}^2)} \geq \left\|v\big|_\Omega\right\|_{C^\alpha(\Omega)}$, we have

$$\|\operatorname{div}(\omega_n Y_n)\|_{C^{\alpha-1}(\Omega)} \leq \left\|f\big|_\Omega\right\|_{C^\alpha(\Omega)} + \left\|v\big|_\Omega\right\|_{C^\alpha(\Omega)} \leq \|f\|_{C^\alpha(\mathbb{R}^2)} + \|v\|_{C^\alpha(\mathbb{R}^2)}.$$

Taking the infimum over all such f and v, we find that

$$\|\operatorname{div}(\omega_n Y_n)\|_{C^{\alpha-1}(\Omega)} \leq \left\|\operatorname{div}\left(\omega_n^* \widetilde{Y_n}\right)\right\|_{C^{\alpha-1}(\mathbb{R}^2)}.$$

By Proposition 9.2 of [BK15], $\left\|\operatorname{div}\left(\omega_n^* \widetilde{Y_n}\right)\right\|_{C^{\alpha-1}(\mathbb{R}^2)} \leq Ce^{\int_0^t \|\nabla u_n^*(s,\cdot)\|_{L^\infty(\mathbb{R}^2)}\, ds}$, so that

$$\|\operatorname{div}(\omega_n Y_n)\|_{C^{\alpha-1}(\Omega)} \leq Ce^{\int_0^t \|\nabla u_n^*(s,\cdot)\|_{L^\infty(\mathbb{R}^2)}\, ds}. \tag{A.2}$$

We now focus on the integrand in the exponent. Since ψ_n solves (2.13), we have that ψ_n is compactly supported in Ω. Thus, we can apply Lemma 2.2.1 to see that

$$\|\psi_n(s,\cdot)\|_{L^p(\Omega)} \leq C\,\|\nabla \psi_n(s,\cdot)\|_{L^p(\Omega)}$$

for any $1 \leq p \leq \infty$. Because the velocities u_n are divergence-free and have a zero normal boundary condition, we can apply Lemmas 2.2.2 and 2.2.3 to see that

$$\|u_n(s,\cdot)\|_{L^p(\Omega)} \leq C\,\|\nabla u_n(s,\cdot)\|_{L^p(\Omega)}$$

for any $1 \leq p \leq \infty$. Using these properties along with Property 2 from Lemma 2.1.4, we have that

$$\begin{aligned}\|\nabla u_n^*(s,\cdot)\|_{L^\infty(\mathbb{R}^2)} &= \left\|\nabla\nabla^\perp \psi_n^*(s,\cdot)\right\|_{L^\infty(\mathbb{R}^2)} \\ &\leq \|\psi_n^*(s,\cdot)\|_{W^{2,\infty}(\mathbb{R}^2)}\end{aligned}$$

$$
\begin{aligned}
&\leq C \left\| \psi_n(s,\cdot) \right\|_{W^{2,\infty}(\Omega)} \\
&= C \left\| \psi_n(s,\cdot) \right\|_{L^\infty(\Omega)} + C \left\| \nabla \psi_n(s,\cdot) \right\|_{L^\infty(\Omega)} + C \left\| \nabla \nabla \psi_n(s,\cdot) \right\|_{L^\infty(\Omega)} \\
&\leq C \left\| \nabla^\perp \psi_n(s,\cdot) \right\|_{L^\infty(\Omega)} + C \left\| \nabla \nabla^\perp \psi_n(s,\cdot) \right\|_{L^\infty(\Omega)} \\
&= C \left\| u_n(s,\cdot) \right\|_{L^\infty(\Omega)} + C \left\| \nabla u_n(s,\cdot) \right\|_{L^\infty(\Omega)} \\
&\leq C \left\| \nabla u_n(s,\cdot) \right\|_{L^\infty(\Omega)}.
\end{aligned}
$$

Combining this with (A.2) gives that

$$
\left\| \operatorname{div} (\omega_n Y_n) \right\|_{C^{\alpha-1}(\Omega)} \leq C e^{\int_0^t C \left\| \nabla u_n(s,\cdot) \right\|_{L^\infty(\Omega)} ds},
$$

as desired. ■

Bibliography

[BC93] A. L. Bertozzi and P. Constantin. Global regularity for vortex patches. *Comm. Math. Phys.*, 152(1):19–28, 1993. 5, 7, 9, 33, 84

[Ber91] Andrea L. Bertozzi. *Existence, uniqueness, and a characterization of solutions to the contour dynamics equation*. PhD dissertation, Princeton University, 1991. 5

[BK15] Hantaek Bae and James P. Kelliher. Propagation of Striated Regularity of Velocity for the Euler Equations. *ArXiv e-prints*, August 2015. 8, 9, 11, 42, 43, 45, 49, 56, 62, 66, 68, 72, 74, 75, 76, 81, 84, 88, 89, 91, 92, 94

[BK21] Hantaek Bae and James P. Kelliher. Propagation of regularity of level sets for a class of active transport equations. *Journal of Mathematical Analysis and Applications*, 497(1):124823, 2021. 9, 11, 74

[But89] Thomas F. Buttke. The observation of singularities in the boundary of patches of constant vorticty. *Phys. Fluids A*, 1(7):1283–1285, 1989. 4

[But90] Thomas F. Buttke. A fast adaptive vortex method for patches of constant vorticity in two dimensions. *J. Comput. Phys.*, 89(1):161–186, 1990. 4

[Che91] Jean-Yves Chemin. Existence globale pour le problème des poches de tourbillon. *C. R. Acad. Sci. Paris Sér. I Math.*, 312(11):803–806, 1991. vi, 5, 8, 11, 72

[Che93] Jean-Yves Chemin. Persistance de structures géométriques dans les fluides incompressibles bidimensionnels. *Ann. Sci. École Norm. Sup. (4)*, 26(4):517–542, 1993. 5, 8, 11, 72

[CJM13] A Crosby, Edward Johnson, and Philip Morrison. Deformation of vortex patches by boundaries. *Physics of Fluids*, 25, 01 2013. 10

[CLM85] Peter Constantin, Peter Lax, and Andrew Majda. A simple one-dimensional model for the three-dimensional vorticity equation. *Comm. Pure Appl. Math.*, 38(6):715–724, 1985. 4

[Dan99] Raphaël Danchin. Persistence of geometrical structures and non-viscous limit for incompressible fluids in any dimension. *Bulletin of the Mathematical Society of France*, 127(2):179–227, 1999. 7, 9

[Dep98] Nicolas Depauw. Poche de tourbillon pour Euler 2D dans un ouvert à bord. In *Journées "Équations aux Dérivées Partielles" (Saint-Jean-de-Monts, 1998)*, pages Exp. No. III, 12. Univ. Nantes, Nantes, 1998. vi, 10, 13, 82

[Dep99] Nicolas Depauw. Poche de tourbillon pour Euler 2D dans un ouvert à bord. *J. Math. Pures Appl. (9)*, 78(3):313–351, 1999. 10, 88

[DM90] D. G. Dritschel and M. E. McIntyre. Does contour dynamics go singular? *Phys. Fluids A*, 2(5):748–753, 1990. 4

[Eva10] Lawrence C. Evans. *Partial differential equations*. American Mathematical Society, Providence, R.I., 2010. 17, 18, 20, 21, 24, 30, 31, 32, 72, 90

[FJ98] F.G. Friedlander and M. Joshi. *Introduction to the Theory of Distributions*. Cambridge University Press, 1998. 62

[Fol13] G.B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2013. 37

[GSR95] Pascal Gamblin and Xavier Saint Raymond. On three-dimensional vortex patches. *Bulletin de la Société Mathématique de France*, 123(3):375–424, 1995. 88

[GT01] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer Berlin Heidelberg, 2001. 76

[GVL13] David Gérard-Varet and Christophe Lacave. The two-dimensional Euler equations on singular domains. *Archive for Rational Mechanics and Analysis*, 30:131–170, 2013. 10

[Kir76] G.R. Kirchhoff. *Vorlesungen uber Matematische Physik*. B.G. Teubner, Leipzig, 1876. 3

[KZ14] Alexander Kiselev and Andrej Zlatos. Blow up for the 2d Euler equation on some bounded domains. *Journal of Differential Equations*, 259, 06 2014. 10

[Lee13] J.M. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer New York, 2013. 12

[Leo09] Giovanni Leoni. *A First Course in Sobolev Spaces*. Graduate studies in mathematics. American Mathematical Soc., 2009. 22

[Maj86] Andrew Majda. Vorticity and the mathematical theory of incompressible fluid flow. *Comm. Pure Appl. Math.*, 39(S, suppl.):S187–S220, 1986. Frontiers of the mathematical sciences: 1985 (New York, 1985). 4

[MB02] A.J. Majda and A.L. Bertozzi. *Vorticity and Incompressible Flow*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2002. 2, 3, 6, 9, 28, 29, 32, 33, 38, 44, 52, 53, 58, 63, 88, 90

[Mey82] R.E. Meyer. *Introduction to Mathematical Fluid Dynamics.* Dover Books on Physics Series. Dover Publications, 1982. 27

[Osg98] W. F. Osgood. Beweis der existenz einer lösung der differentialgleichung $\frac{dy}{dx} = f(x, y)$ ohne hinzunahme der cauchy-lipschitz'schen bedingung. *Monatshefte für Mathematik und Physik*, 9(1):331–345, 1898. 90

[Ser92] Philippe Serfati. *Étude mathématique de flammes infiniment minces en combustion. Résultats de structure et de régularité pour l'équation d'Euler incompressible.* PhD thesis, Thèse de doctorat dirigée par Raviart, Pierre-Arnaud Mathématiques Paris, 1992. 8, 74

[Ser94a] Philippe Serfati. Régularité stratifiée et équation d'Euler 3d à temps grand. *Comptes Rendus de l Académie des Sciences - Series I - Mathematics*, 01 1994. 8, 74

[Ser94b] Philippe Serfati. Une preuve directe d'existence globale des vortex patches 2D. *C. R. Acad. Sci. Paris Sér. I Math.*, 318(6):515–518, 1994. vi, 8, 74

[Ste70] E.M. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, 1970. 20

[Str08] Walter A. Strauss. *Partial Differential Equations : An Introduction.* Wiley, second edition, December 2008. 33

[Wol33] W. Wolibner. Un theorème sur l'existence du mouvement plan d'un fluide parfait, homogene, incompressible, pendant un temps infiniment long. *Mathematische Zeitschrift*, 37:698–726, 1933. 30, 50

[Yud63] V. I. Yudovich. Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 3:1032–1066, 1963. 3, 9, 30

[ZHR79] Norman J. Zabusky, M. H. Hughes, and K. V. Roberts. Contour dynamics for the Euler equations in two dimensions. *J. Comput. Phys.*, 30(1):96–106, 1979. 3, 6