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Authors

Jin, Wen-Long

Gan, Qi-Jian

Lebacque, Jean-Patrick

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A kinematic wave theory of capacity drop

Wen-Long Jin ^{*}, Qi-Jian Gan [†] and Jean-Patrick Lebacque [‡]

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Abstract

Capacity drop at active bottlenecks is one of the most puzzling traffic phenomena, but a thorough understanding of its mechanism is critical for designing variable speed limit and ramp metering strategies. In this study, within the framework of the kinematic wave theory, we propose a simple model of capacity drop based on the observation that capacity drop occurs when an upstream queue forms at an active bottleneck. Different from existing models, the new model still uses continuous fundamental diagrams but employs an entropy condition defined by a discontinuous boundary flux function, which introduces a traffic state-dependent capacity constraint. For a lane-drop area, we demonstrate that the model is well-defined, and its Riemann problem can be uniquely solved. After deriving the flow-density relations upstream and downstream to a bottleneck location, we find that the model can replicate the following three characteristics of capacity drop: the maximum discharge flow-rate can be reached only when both upstream and downstream traffic conditions are uncongested, capacity drop occurs when the bottleneck is activated, and some steady traffic states cannot be observed at both locations. We show that the new model is bistable subject to perturbations in initial and boundary conditions. With empirical observations at a merging bottleneck we also verify the three characteristics of capacity drop predicted by the new model. Through this study, we establish that the new model is physically meaningful, conceptually simple, computationally efficient, and mathematically tractable. We finally discuss future extensions and potential applications of the new model.

Keywords: Capacity drop characteristics; Kinematic wave theory; Continuous fundamental diagram; Discontinuous entropy condition; Riemann problem; Stability.

^{*}Department of Civil and Environmental Engineering, California Institute for Telecommunications and Information Technology, Institute of Transportation Studies, 4000 Anteater Instruction and Research Bldg, University of California, Irvine, CA 92697-3600. Tel: 949-824-1672. Fax: 949-824-8385. Email: wjin@uci.edu. Corresponding author

[†]Department of Civil and Environmental Engineering, Institute of Transportation Studies, University of California, Irvine, CA 92697-3600. Email: qgan@uci.edu

[‡]Université Paris-Est, IFSTTAR, GRETTIA, 14-20 Boulevard Newton, Cité Descartes, Champs sur Marne, 77447 Marne la Vallée Cedex 2, France. Email: jean-patrick.lebacque@ifsttar.fr

1 Introduction

Since the 1990s, the so-called two-capacity or capacity-drop phenomenon of active bottlenecks, in which “maximum flow rates decrease when queues form”, has been observed and verified at many bottleneck locations (Banks, 1990, 1991b; Hall and Agyemang-Duah, 1991). For example, at a merge bottleneck, when the total demand of the upstream mainline freeway and the on-ramp exceeds the capacity of the downstream mainline freeway, a queue forms on the mainline freeway, and the discharge flow-rate drops below the capacity of the downstream mainline freeway. Such “capacity drop” has also been observed at tunnels, lane drops, curves, and upgrades, where the bottlenecks cannot provide sufficient space for upstream vehicles (Cassidy and Bertini, 1999; Chung et al., 2007). Capacity drop also occurs at bottlenecks caused by work zones (Krammes and Lopez, 1994; Dixon et al., 1996; Jiang, 1999) as well as accidents/incidents (Smith et al., 2003).

A drop in the downstream bottleneck’s discharge flow-rate can further reduce the discharge flow-rates of impacted upstream off-ramps and the total discharge flow-rate of the whole corridor and, therefore, prolong vehicles’ travel times (Newell, 1993; Daganzo, 1999). That the capacity of a road network may drop substantially when it is most needed during the peak period has been a baffling feature of freeway traffic dynamics (Papageorgiou and Kotsialos, 2002). Hence to prevent or delay the occurrence of capacity drop has been an important motivation and theoretical foundation for developing ramp metering, variable speed limits, and other control strategies (Banks, 1991a; Papageorgiou et al., 1991, 1997; Cassidy and Rudjanakanoknad, 2005; Papageorgiou et al., 2005, 2007).

Since 1960s, it has been observed that the flow-density relation, i.e., the fundamental diagram, can be a discontinuous function or multi-valued with a reverse-lambda shape (Edie, 1961; Drake et al., 1967; Koshi et al., 1983; Payne, 1984; Hall et al., 1992). This is different from traditional fundamental diagrams derived from car-following models in steady states, in which the flow-rate is a continuous function of the density. In (Hall and Agyemang-Duah, 1991; Hall et al., 1992), it was shown that discontinuous fundamental diagrams generally arise inside the bottleneck area and suggested that the discontinuity is associated with the capacity drop phenomenon. In the literature, many models of capacity drop have been based on the assumption of discontinuous fundamental diagrams. For example, in (Lu et al., 2008, 2009), an attempt was made to describe capacity drop with discontinuous fundamental diagrams within the framework of the LWR model (Lighthill and Whitham, 1955; Richards, 1956).

However, a discontinuous fundamental diagram is challenged both theoretically and empirically. Theoretically, a discontinuous flow-density relation is non-differentiable at the discontinuous point (usually the critical density) and leads to infinite characteristic wave speeds (Li and Zhang, 2013). Clearly this contradicts the fact that information travels at a finite speed along a traffic stream. Empirically, even though many studies confirm the existence of discontinuous fundamental diagrams inside a bottleneck area, e.g., Figure 4 of (Hall et al., 1992), Cassidy (1998) demonstrated that, in near-stationary states, bivariate fundamental diagrams are still continuous at a location upstream to a bottleneck with capacity drop, but densities in some ranges cannot be observed.

Kinematic wave theories, e.g., the LWR model and the Cell Transmission Model (CTM) (Daganzo, 1995), have been powerful tools to analyze and simulate the queue formation, propagation,

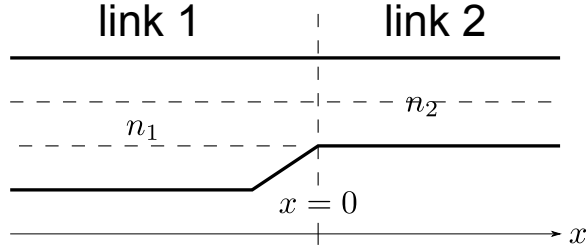


Figure 1: A lane-drop bottleneck

and dissipation processes through shock and rarefaction waves connecting different steady states. They have been widely used in designing ramp metering and other control strategies (Gomes and Horowitz, 2006). To the best of our knowledge, however, there has been no systematic theory of capacity drop with continuous fundamental diagrams.

In this study we propose a new model of capacity drop to reconcile continuous fundamental diagrams with capacity drop. For an active lane-drop bottleneck, as shown in Figure 1, we attempt to replicate the observation that ‘‘maximum flow rates decrease when queues form’’ with the continuous CTM formulation of the kinematic wave theory developed in (Jin et al., 2009; Jin, 2012b), in which the junction flux function in terms of upstream demands and downstream supplies is used as an entropy condition to pick out unique, physical solutions. In particular, from CTM we can see that an upstream queue forms when the upstream demand is larger than the downstream supply. Then we introduce a new flux function based on the observation that upstream congestion and capacity drop occur immediately after the upstream demand exceeds the downstream supply. Here the new flux function is a discontinuous function in upstream demand and downstream supply. This is different from traditional flux functions, which are generally continuous (Daganzo, 1995; Lebacque, 1996; Jin and Zhang, 2003b; Ni and Leonard, 2005; Lebacque and Khoshyaran, 2005; Jin, 2010; Tampère et al., 2011; Jin, 2012b). With the new model we aim to reproduce the following characteristics of capacity drop: (i) when both upstream and downstream locations are uncongested, the discharge flow-rate can reach the downstream capacity; (ii) capacity drop occurs when the bottleneck is activated; i.e., when the upstream location is congested but downstream not, the discharge flow-rate drops below the downstream capacity; and (iii) some steady traffic states cannot be observed at both upstream and downstream locations, and the observed flow-density relations are discontinuous.

In the literature, there have been many studies on capacity drop. This study has a number of distinctive features.

- In contrast to existing kinematic wave models of capacity drop (Lu et al., 2008, 2009), the new model still uses continuous fundamental diagrams for the upstream and downstream links and is therefore devoid of unrealistic infinite information propagation speeds.¹ However,

¹A characteristic wave, whose speed equals the derivative of flow-rate in density, can be considered as the information propagation wave of a small disturbance around a constant traffic density. Thus a discontinuous fundamental diagram leads to an infinite information propagation speed around the density where the flow-rate jumps.

discontinuous flow-density relation can arise inside the bottleneck area where we can observe mixed congested and uncongested traffic states.

- The new model is still of the first order as the LWR model. In (Carlson et al., 2010; Parzani and Buisson, 2012), higher-order continuum models were shown to replicate capacity drop, but the capacity in higher-order models may be different from the generally used value in steady states (Zhang, 2001).
- The new model is phenomenological, different from (Leclercq et al., 2011), where the capacity drop magnitude was endogenously calculated by considering merging vehicles as moving bottlenecks. We assume that the magnitude of capacity drop is given or calibrated for a bottleneck. Such a model is much simpler and more suitable for system-level control and management applications.
- The new model is conceptually simple, computationally efficient, and mathematically tractable. In particular, the new junction flux function can be readily incorporated in various models of network traffic flow, including CTM, Link Transmission Model (Yperman et al., 2006; Yperman, 2007; Jin, 2015), and Link Queue Model (Jin, 2012c), to analyze and simulate impacts of capacity drop on traffic dynamics. Therefore, the new model is quite useful for analyzing and developing new traffic control strategies.

The rest of the paper is organized as follows. In Section 2, we present a new model for capacity drop at a lane-drop bottleneck and demonstrate that it is well-defined under Riemann initial conditions. In Section 3, we discuss the analytical properties of the new model and demonstrate the model replicates the three characteristics of capacity drop. In Section 4, we present an empirical study to validate the new model. In Section 5, we make some concluding remarks.

2 A kinematic wave model of capacity drop at a lane-drop bottleneck

For a road with a lane-drop bottleneck, shown in Figure 1, the upstream link 1 has n_1 lanes and the downstream link 2 has $n_2 < n_1$ lanes. For the purpose of simple analyses, we omit the dynamics inside the transition region from n_1 lanes to n_2 lanes and assume that the lane-drop bottleneck is at $x = 0$. In reality, the exact bottleneck location can vary from time to time.

We denote total traffic density, speed, and flow-rate by $k(x, t)$, $v(x, t)$, $q(x, t)$ respectively, which are all functions of location x and time t . The number of lanes at x is denoted by $n(x)$. Hereafter we will omit (x, t) from these variables unless necessary. Then the LWR model of traffic flow on the road shown in Figure 1 can be defined by the following rules:

R1. The constitutive law of continuous media: $q = kv$.

R2. The location-dependent fundamental diagram (Greenshields, 1935): $v = V(n, k)$ and $q = kV(n, k) = Q(n, k)$.

R3. The continuity equation: $\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0$.

R4. The existence of weak solutions: discontinuous shock waves can develop from continuous initial conditions.

R5. The entropy condition: unique, physical solutions of the LWR model should satisfy an entropy condition.

The first three rules lead to the following inhomogeneous LWR model

$$\frac{\partial k}{\partial t} + \frac{\partial Q(n, k)}{\partial x} = 0, \quad (1)$$

which is a non-strictly hyperbolic conservation law (Temple, 1982; Isaacson and Temple, 1992). Among the five rules, R1, R3, and R4 are generic for all continuum dynamics, but R2 and R5 are system specific. For a traffic system, R2 is determined by static characteristics, i.e. flow- and speed-density relations in steady states, and R5 by dynamic car-following, lane-changing, merging, diverging, and other driving behaviors.

In (Jin and Zhang, 2003a), the inhomogeneous LWR model (1) was solved as a resonant nonlinear system. In (Daganzo, 2006), a variational principle was proposed to uniquely solve (1). In (Jin et al., 2009), it was shown that the boundary flux function, which was initially introduced in CTM (Daganzo, 1995), can be used as an entropy condition. In these studies, fundamental diagrams are continuous on all links, but capacity drop has not been modeled within the framework.

In this study, we still model capacity drop at a lane-drop bottleneck with the inhomogeneous LWR model, (1). Here we still employ continuous fundamental diagrams for both the upstream and downstream links. An example is the following triangular fundamental diagram, which has been derived from car-following models and verified by observations (Munjal et al., 1971; Haberman, 1977; Newell, 1993):

$$Q(n, k) = \min \left\{ v^* k, \frac{1}{\tau} \left(n - \frac{k}{k^*} \right) \right\}, \quad (2)$$

where v^* is the free-flow speed, τ the time-gap, k^* the jam density per lane, and $k_c(n) = \frac{nk^*}{1+\tau v^* k^*}$ the total critical density of n lanes. Then the total capacity is $C(n) = Q(n, k_c(n))$. A traffic state is called strictly under-critical (SUC), critical (C), or strictly over-critical (SOC) if $k <, =,$ or $> k_c(n)$, respectively. A UC state can be either SUC or C, and an OC state can be either SOC or C. In this fundamental diagram, the flow-rate is a continuous function in the density, but the relation varies with the number of lanes. With the triangular fundamental diagram, the characteristic wave speed Q_k equals v_f or $-\frac{1}{\tau k^*}$, which is always finite. Thus the new model of capacity drop is devoid of infinite information propagation speeds in reverse-lambda like discontinuous fundamental diagrams.

2.1 A discontinuous entropy condition

Then, following the kinematic wave theory in (Jin et al., 2009), we introduce a discontinuous boundary flux function in upstream demands and downstream supplies as an entropy condition. But

this model is phenomenological and approximate since (i) the capacity drop magnitude is given, and (ii) capacity drop occurs immediately following the upstream congestion and exactly at the lane-drop location, $x = 0$.

We denote traffic demand and supply at (x, t) by $d(x, t)$ and $s(x, t)$, respectively. For a continuous flow-density relation $Q(n, k)$, which is unimodal in k , traffic demand and supply are respectively its increasing and decreasing branches (Engquist and Osher, 1980; Daganzo, 1995; Lebacque, 1996):

$$\begin{aligned} d &= D(n, k) \equiv Q(n, \min\{k_c(n), k\}), \\ s &= S(n, k) \equiv Q(n, \max\{k_c(n), k\}), \end{aligned}$$

where $k_c(n)$ is the critical density for n lanes. Since $Q(n, k)$ is unimodal in k , $D(n, k)/S(n, k)$ is a strictly increasing function in k . If we define the congestion level by $\gamma = d/s$, then traffic density is a function of γ

$$k = K(n, \gamma), \quad (3)$$

such that $D(n, k)/S(n, k) = \gamma$. In this sense, a traffic state can be uniquely determined by the demand and supply pair. In addition, a traffic state is SUC, C, or SOC if $d < s = C(n)$, $d = s = C(n)$, and $s < d = C(n)$, respectively.

Based on the definitions of traffic demand and supply, in (Jin et al., 2009) it was shown that the following flux function is a valid entropy condition for the inhomogeneous LWR model (1):

$$q(x, t) = \min\{d(x^-, t), s(x^+, t)\}, \quad (4)$$

where $d(x^-, t)$ and $s(x^+, t)$ are the upstream demand and downstream supply, respectively, at x . That is, the LWR model, (1), coupled with (4) has unique weak solutions with given initial and boundary conditions (Holden and Risebro, 1995). In addition, (4) is consistent with the traditional entropy conditions by (Lax, 1972), (Ansorge, 1990), and (Isaacson and Temple, 1992). From (4) we can see that a queue forms on the upstream link when the upstream demand is greater than the downstream supply. Furthermore, if the downstream link is uncongested; i.e., when the lane-drop area is an active bottleneck, the maximum throughput of the lane-drop bottleneck is the capacity of the downstream link: $q(0, t) = C_2 = Q(n_2, k_c(n_2))$. Therefore (4) cannot model the capacity drop phenomenon.²

Since capacity drop arises with a queue on the upstream link 1, it is associated with the traffic dynamics in the transition region around $x = 0$ between the two links, and it is reasonable to modify the entropy condition, (4), to capture this dynamic feature. That is, capacity drop is triggered when the upstream demand is greater than the downstream supply and there is an upstream queue. Thus we still apply (4) as the entropy condition for traffic inside the upstream link 1 and downstream link 2, but introduce the following new entropy condition for the transition region at $x = 0$:

$$q(0, t) = \begin{cases} d(0^-, t), & d(0^-, t) \leq s(0^+, t) \\ \min\{s(0^+, t), C_*\}, & d(0^-, t) > s(0^+, t) \end{cases} \quad (5)$$

²In (Jin, 2013), it was shown that systematic lane changes can reduce C_2 , which can be computed from the number of lanes, n_1 and n_2 , the average duration of each lane change, and the length of the lane-changing region. However, such a capacity reduction phenomenon is different from capacity drop, as they have different features; i.e., (4) with capacity reduction caused by lane changes cannot capture the aforementioned three characteristics of capacity drop.

where $d(0^-, t)$ is the upstream demand, $s(0^+, t)$ the downstream supply, and C_* the dropped capacity. Here we assume that $C_* < C_2$, and the capacity-drop ratio is defined by

$$\Delta = 1 - \frac{C_*}{C_2}.$$

Based on the observation that the maximum flow-rate for the bottlenecks can reach 2300 vphpl in free-flow traffic (Federal highway administration, 1985; Hall and Agyemang-Duah, 1991), capacity drop magnitudes have been quantified for different locations. Generally, the magnitude of capacity drop is in the order of 10%, even 20% (Persaud et al., 1998; Cassidy and Bertini, 1999; Bertini and Leal, 2005; Chung et al., 2007), and such a drop is stable, although interactions among several bottlenecks can cause fluctuations in discharge flow-rates (Kim and Cassidy, 2012).

Theorem 2.1 *The junction model in (4) is continuous in both the upstream demand $d(0^-, t)$ and the downstream supply $s(0^+, t)$. But the junction model in (5) is discontinuous when $\Delta > 0$; i.e., the boundary flux $q(0, t)$ is a discontinuous function in both the upstream demand $d(0^-, t)$ and the downstream supply $s(0^+, t)$.*

Proof. It is straightforward to show that the junction model in (4) is continuous in both the upstream demand $d(0^-, t)$ and the downstream supply $s(0^+, t)$.

For a given downstream supply $s(0^+, t) > C_*$, if we increase the upstream demand $d(0^-, t)$ from 0 to $C_1 > s(0^+, t)$, then from (5) the junction flux is given by

$$q(0, t) = \begin{cases} d(0^-, t), & d(0^-, t) \leq s(0^+, t) \\ C_*, & d(0^-, t) > s(0^+, t) > C_* \end{cases}$$

which jumps at $d(0^-, t) = s(0^+, t)$. Similarly, for a given upstream demand $C_2 > d(0^-, t) > C_*$, if we increase the downstream supply $s(0^+, t)$ from 0 to C_2 , then from (5) the junction flux is given by

$$q(0, t) = \begin{cases} C_*, & s(0^+, t) < d(0^-, t) \\ d(0^-, t), & s(0^+, t) \geq d(0^-, t) \end{cases}$$

which jumps at $s(0^+, t) = d(0^-, t)$. Therefore, the junction model in (5) is a discontinuous function in both the upstream demand and downstream supply. ■

If we introduce an indicator function, $I_{d(0^-, t) > s(0^+, t)}$, which equals 1 if $d(0^-, t) > s(0^+, t)$ and 0 otherwise, then (5) can be re-written as

$$q(0, t) = \min\{d(0^-, t), s(0^+, t), C_2(1 - \Delta \cdot I_{d(0^-, t) > s(0^+, t)})\}.$$

We can see that the new flux function, i.e., entropy condition, is consistent with the following macroscopic rules:

1. The flux is maximized: $\max q(0, t)$.
2. The flux is not greater than the upstream demand or the downstream supply: $q(0, t) \leq d(0^-, t)$, and $q(0, t) \leq s(0^+, t)$.

3. When the upstream link is congested; i.e., when the upstream demand is higher than the downstream supply, the flux is not greater than the dropped capacity: $q(0,t) \leq C_2(1 - \Delta \cdot I_{d(0^-,t) > s(0^+,t)})$.

Therefore the new entropy condition is equivalent to the following optimization problem:

$$\max q(0,t), \tag{6}$$

s.t.,

$$\begin{aligned} q(0,t) &\leq d(0^-,t), \\ q(0,t) &\leq s(0^+,t), \\ q(0,t) &\leq C_*, \text{ when } d(0^-,t) > s(0^+,t). \end{aligned}$$

Thus we obtain a new LWR model with capacity-drop effect: (1) with (4) at $x \neq 0$ and (5) at $x = 0$. The model differs from the traditional LWR model only in the entropy condition at $x = 0$. We have the following observations regarding the boundary flux function in (5):

1. When the upstream demand is not greater than the downstream supply, (5) is consistent with (4), and the new LWR model has the same kinematic wave solutions as the traditional one.
2. However, when the upstream demand is greater than the downstream supply and the downstream supply is greater than C_* , capacity drop occurs, and the discharge flow-rate is bounded by C_* .
3. The flux function (5) is discontinuous in both the upstream demand and the downstream supply. This is different from many existing flux functions used in CTM (Tampère et al., 2011; Jin, 2012b).
4. The new LWR model with (5) is purely phenomenological with an exogenous parameter, C_* or Δ , and the driving behaviors and related mechanisms for capacity drop cannot be explained by the model. The model can only be used to describe kinematic waves caused by capacity drop at the lane-drop bottleneck.

2.2 The Riemann problem

In this subsection, we show that the LWR model (1) is still well-defined with the new entropy condition (5) at $x = 0$ by demonstrating that the Riemann problem has a unique solution with the following initial condition:

$$k(x,0) = \begin{cases} k_1, & x < 0; \\ k_2, & x \geq 0. \end{cases}$$

As for other systems of hyperbolic conservation laws, solutions to the Riemann problem for (1) at the capacity-drop bottleneck are of physical, analytical, and numerical importance: physically, they

can be used to analyze traffic dynamics caused by capacity drop; analytically, (1) is well-defined if and only if the Riemann problem is uniquely solved Bressan and Jenssen (2000); and numerically, they can be incorporated into the Cell Transmission Model Daganzo (1995); Lebacque (1996).

Here we solve the Riemann problem by following the analytical framework in (Jin et al., 2009; Jin, 2012a): (i) the problem is solved in the demand-supply space, with initial conditions:

$$U(x,0) = \begin{cases} (d_1, s_1), & x < 0; \\ (d_2, s_2), & x \geq 0. \end{cases} \quad (7)$$

(ii) in the Riemann solutions on each link, a stationary state arises on a link along with a shock or rarefaction wave, which connects the stationary state and the initial state and is determined by a separate Riemann problem for the homogeneous LWR model; (iii) the stationary state should be inside a feasible domain, such that the shock or rarefaction wave propagates backward on the upstream link 1 and forward on the downstream link 2, and the boundary flux $q(0,t)$ equals the stationary flow-rate; (iv) the weak solution space is enlarged to include a filmy interior state on each link at $x = 0$, which occupies no space (of measure zero); (v) the entropy condition, (5) or (6), is applied on the interior states; and (vi) we prove that the stationary states and, therefore, the Riemann problem are uniquely solved.

In the demand-supply space, the initial conditions on the upstream and downstream links are denoted by $U_1 = (d_1, s_1)$ and $U_2 = (d_2, s_2)$, respectively, where $d_i = D(n_i, k_i)$ and $s_i = S(n_i, k_i)$ for $i = 1, 2$. In the Riemann solutions, upstream stationary and interior states are $U_1^* = (d_1^*, s_1^*)$ and $U_1^0 = (d_1^0, s_1^0)$ respectively, and downstream stationary and interior states are $U_2^* = (d_2^*, s_2^*)$ and $U_2^0 = (d_2^0, s_2^0)$ respectively. Then the kinematic waves on upstream and downstream links are determined by $RP(U_1, U_1^*)$ and $RP(U_2^*, U_2)$ respectively, which are the Riemann problems for the traditional, homogeneous LWR model. That is, $RP(U_1, U_1^*)$ is the Riemann problem for $\frac{\partial k}{\partial t} + \frac{\partial Q(n_1, k)}{\partial x} = 0$ with $k(x, 0) = \begin{cases} k_1, & x < 0 \\ k_1^*, & x > 0 \end{cases}$, where $k_1^* = K(n_1, d_1^*/s_1^*)$, and with the traditional Lax entropy condition based on characteristics or the entropy condition in (4). Since kinematic wave speeds of $RP(U_1, U_1^*)$, $RP(U_1^*, U_1^0)$, $RP(U_2^0, U_2^*)$, $RP(U_2^*, U_2)$ have to be non-positive, positive, negative, and non-negative, respectively, we have the following feasible stationary and interior states (Jin et al., 2009):

1. The upstream stationary state is SOC, if and only if $q < d_1$ and $U_1^* = U_1^0 = (C_1, q)$; it is UC iff $q = d_1$, $U_1^* = (q, C_1)$, and $d_1^0 > d_1$.
2. The downstream stationary state is SUC if and only if $q < s_2$ and $U_2^* = U_2^0 = (q, C_2)$; it is OC iff $q = s_2$, $U_2^* = (C_2, q)$, and $d_2^0 > s_2$.

We use (6) as an entropy condition in interior states:

$$\max_{U_1^*, U_2^*} q, \quad (8)$$

s.t.

$$\begin{aligned} q &\leq d_1^0, \\ q &\leq s_2^0, \\ q &\leq C_*, \text{ if } d_1^0 > s_2^0. \end{aligned}$$

The solution of the optimization problem is given by

$$q = \begin{cases} d_1^0, & d_1^0 \leq s_2^0 \\ \min\{s_2^0, C_*\}, & d_1^0 > s_2^0 \end{cases} \quad (9)$$

which is consistent with (5).

In the following theorem, we show that the stationary states are uniquely solved with (8). Furthermore, since one can calculate the boundary flux and the shock or rarefaction waves on both links from the unique stationary states, the Riemann problem is uniquely solved.

Theorem 2.2 *For the Riemann problem of (1) with (4) at $x \neq 0$ and (6) at $x = 0$, the stationary states U_1^* and U_2^* and, therefore, the kinematic waves on links 1 and 2 exist and are unique. That is, the optimization problem (8) has a unique solution in q , U_1^* , and U_2^* . In particular,*

$$q = \begin{cases} d_1, & d_1 \leq s_2 \\ \min\{s_2, C_*\}, & d_1 > s_2 \end{cases}, \quad (10)$$

which is the same as (5). Therefore, the new flux function (5) is invariant in the sense of (Lebacque and Khoshyaran, 2005; Jin, 2012a).

The proof of Theorem 2.2 is given in Appendix A. Similar to the inhomogeneous LWR model without capacity drop, the capacity-drop model can have two waves on the two links simultaneously; in contrast, the homogeneous LWR model can only have one wave solution for the Riemann problem. However, the capacity-drop model with (6) at $x = 0$ is different from the non-capacity-drop model with (4) at $x = 0$ when $C_* < s_2 \leq C_2$ and $s_2 < d_1 \leq C_1$: in the capacity drop model, $q = C_*$, $U_1^* = (C_1, C_*)$, $U_2^* = (C_*, C_2)$, a backward shock or rarefaction wave forms on the upstream link, and a forward shock or rarefaction wave forms on the downstream link; but in the non-capacity drop model, $q = s_2$, $U_1^* = (C_1, s_2)$, $U_2^* = (C_2, s_2)$, a backward shock or rarefaction wave forms on the upstream link, and a forward rarefaction or no wave forms on the downstream link. That is, when capacity drop occurs, the flow-rate is dropped, and the downstream traffic becomes strictly under-critical.

Consider the example shown in Figure 2, where the initial upstream and downstream states are at A and B , respectively, $C_* < s_2 < C_2$, and $s_2 < d_1 < C_1$. In solutions to the capacity-drop model shown in Figure 2(a), the stationary states on the upstream and downstream links become A' and B' , respectively; the boundary flux becomes C_* , which is smaller than the flow-rate of B ; a backward shock wave forms on the upstream link, and a forward shock wave forms on the downstream link. In solutions to the model without capacity drop shown in Figure 2(b), the stationary state on the upstream link becomes A'' , but the stationary state on the downstream link is the same as the initial state B ; the boundary flux equals the flow-rate of B ; a backward shock wave forms on the upstream link, but there is no wave on the downstream link.

3 Analytical properties of the new capacity drop model

In this section, we analyze properties of the LWR model (1) with the discontinuous entropy condition (5) at the lane-drop bottleneck. We show that the model replicates the three characteristics of capacity drop and that the model is bistable.

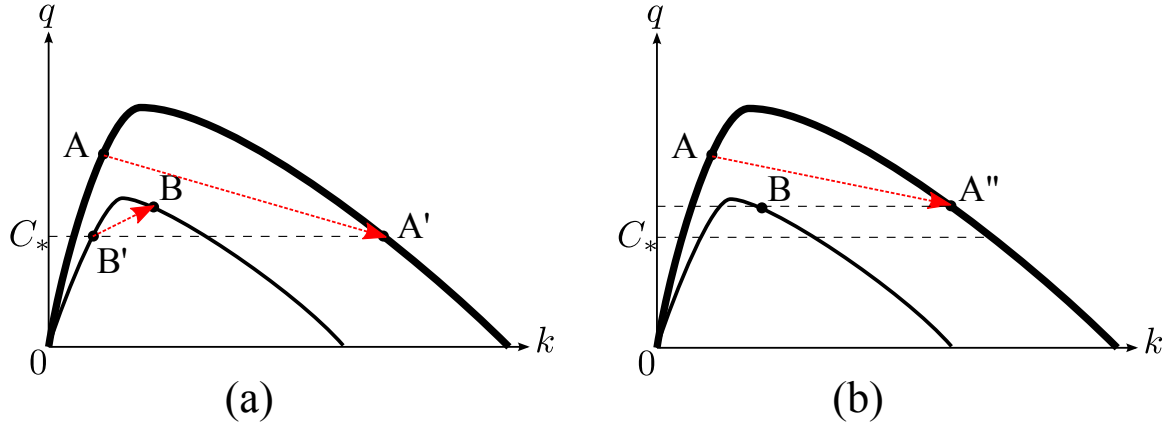


Figure 2: Kinematic wave solutions of (1) with an initial upstream condition at A and an initial downstream condition at B: (a) with capacity drop; (b) without capacity drop

3.1 Upstream and downstream flow-density relations in steady states

In this subsection, we consider the following traffic statics problem on a road section $x \in [-X, Y]$ with a lane-drop at $x = 0$. Initially the road section is empty with $k(x, 0) = 0$. The upstream demand is constant, $d(-X^-, t) = d_0$, and the downstream supply is also constant, $s(Y^+, t) = s_0$. We are interested in finding stationary states in the road network (Jin, 2012d).

In steady states, both the upstream and downstream links carry uniform traffic³, and we assume that their densities are k_1 and k_2 , respectively. Then the corresponding demands and supplies are (d_1, s_1) and (d_2, s_2) , respectively. We denote the flow-rate in the network by q , which is constant at all locations. Then using (5) at the lane-drop location and (4) at the origin and destination, we have

$$q = \min\{d_0, s_1\}, \quad (11a)$$

$$q = \begin{cases} d_1, & d_1 \leq s_2 \\ \min\{s_2, C_*\}, & d_1 > s_2 \end{cases} \quad (11b)$$

$$q = \min\{d_2, s_0\}. \quad (11c)$$

In addition, from the definitions of supply and demand we have

$$C_1 = \max\{d_1, s_1\}, \quad (11d)$$

$$C_2 = \max\{d_2, s_2\}. \quad (11e)$$

From the five equations above and the evolution of traffic dynamics⁴, we can find the following steady-state solutions of (d_1, s_1) and (d_2, s_2) in three traffic regimes:⁵

³Here we do not consider stationary states with zero-speed shock waves on a link as in (Jin, 2012d).

⁴The evolution of traffic dynamics can be analyzed with shock and rarefaction waves, but the detailed analysis is omitted.

⁵Without loss of generality, we assume that $d_0 \leq C_1$ and $s_0 \leq C_2$.

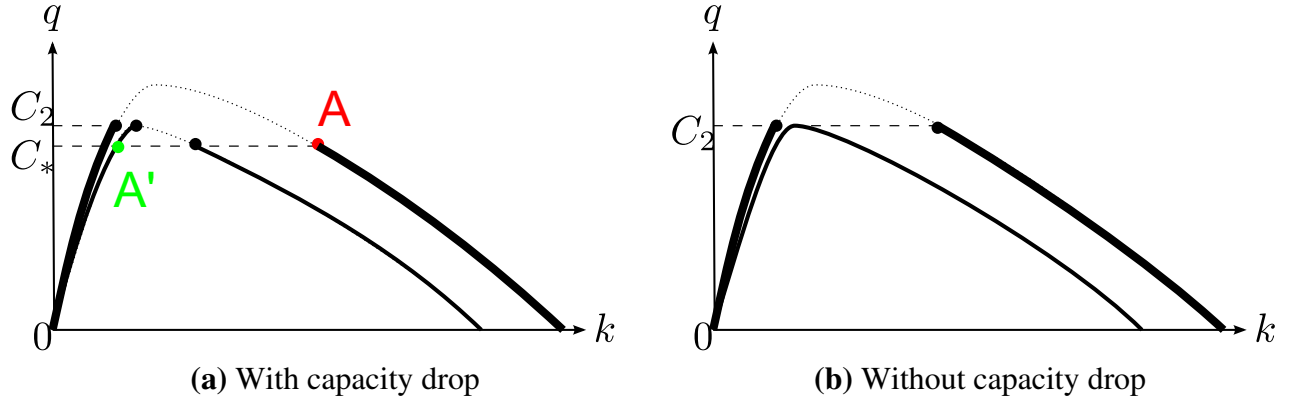


Figure 3: Stationary flow-density relations of upstream and downstream links in a lane drop area: thinner solid curves for the downstream link, thicker solid curves for the upstream link, and steady traffic states on the dotted curves cannot be observed

SUC-UC. When $d_0 \leq s_0 \leq C_2$, $q = d_0$, $(d_1, s_1) = (d_0, C_1)$, and $(d_2, s_2) = (d_0, C_2)$. In this case, both links are uncongested, and the bottleneck is not activated.

SOC-SOC. When $d_0 > s_0$ and $s_0 \leq C_*$, $q = s_0$, $(d_1, s_1) = (C_1, q)$, and $(d_2, s_2) = (C_2, q)$. In this case, both links are congested, and congestion propagates from the downstream link.

SOC-SUC. When $d_0 > s_0$ and $s_0 > C_*$, $q = C_*$, $(d_1, s_1) = (C_1, q)$, and $(d_2, s_2) = (q, C_2)$. In this case, link 1 is congested, but link 2 not. That is, the lane-drop bottleneck is activated, and capacity drop occurs.

Then from the relationship between congestion level and density, we can find corresponding densities and therefore fundamental diagrams in stationary states on both upstream and downstream links, shown in Figure 3(a): the flow-density relations in the SUC-UC regime are given by the left curves; the flow-density relations in the SOC-SOC regime are given by the right curves; in the SOC-SUC regime, the upstream link is at the congested state A, and the downstream link at the uncongested state A'.

From Figure 3(a) we have the following observations: (i) The maximum discharge flow-rate occurs in the SUC-UC regime and equals the downstream capacity, C_2 . (ii) The bottleneck is activated in the SOC-SUC regime, and the discharge flow-rate is at the reduced rate of C_* . (iii) Traffic states on the dotted curves cannot be observed, even though the original fundamental diagrams are continuous. The difference between the two flow-density relations also highlights the importance of the locations where data are collected (Hall and Agyemang-Duah, 1991).

By comparison, the flow-density relations in steady states without capacity drop are shown in Figure 3(b), from which we can also observe three regimes:

SUC-UC. When $d_0 \leq s_0 \leq C_2$, $q = d_0$, $(d_1, s_1) = (d_0, C_1)$, and $(d_2, s_2) = (d_0, C_2)$. In this case, both links are uncongested, and the bottleneck is not activated.

SOC-SOC. When $d_0 > s_0$ and $s_0 < C_2$, $q = s_0$, $(d_1, s_1) = (C_1, q)$, and $(d_2, s_2) = (C_2, q)$. In this case, both links are congested, and congestion propagates from the downstream link.

SOC-C. When $d_0 > s_0$ and $s_0 = C_2$, $q = C_2$, $(d_1, s_1) = (C_1, q)$, and $(d_2, s_2) = (C_2, C_2)$. In this case, the bottleneck is activated, but there exists no capacity drop.

In addition, steady traffic states with $q > C_2$ cannot be observed on the upstream link. Therefore, we can see that the new entropy condition, (6), is necessary and sufficient to replicate the capacity drop phenomenon. However, it is not necessary for unobservable traffic states.

Note that the existence of unobservable traffic states may attribute to the hypothesis of ‘‘discontinuous’’ fundamental diagrams in the literature, e.g., Figure 15 of (Drake et al., 1967), and Figure 6 for shoulder lane in (Hall et al., 1986). In these figures, flow-density relations were approximated by discontinuous fundamental diagrams, but can also be approximated by continuous fundamental diagrams with unobserved traffic states.

3.2 Stability subject to perturbations in initial and boundary conditions

In this subsection, we study the stability of the capacity-drop model, (1) with (5) at $x = 0$, subject to perturbations to initial conditions. In particular, we consider solutions of the following perturbed Riemann problem (Liu, 1987; Mascia and Sinestrari, 1997):

$$k(x, 0) = \begin{cases} k_1, & x < -L \\ k_0, & -L < x < 0 \\ k_2, & x > 0 \end{cases} \quad (12)$$

where a perturbation k_0 is applied on the upstream road section between $-L$ and 0 . We expect that results will be similar if we apply a perturbation on the downstream link. Note that the LWR model (1) with entropy condition (4) is always stable with respect to perturbations to initial conditions.

We denote the demand and supply corresponding to k_i by (d_i, s_i) ($i = 0, 1, 2$). One can show that, when $d_1 < \min\{C_*, s_2\}$ or $d_1 > s_2$, solutions with initial condition (12) are the same as those with initial condition (7) at a large time $t > 0$. In the long run, capacity drop always occurs when $d_1 > s_2$ and does not occur when $d_1 < \min\{C_*, s_2\}$, whether there is perturbation or not. That is, under these initial conditions, the LWR model (1) with entropy condition (5) is stable subject to perturbations k_0 .

However, as shown in Figure 4, when $C_* < d_1 \leq s_2$, solutions to the perturbed Riemann problem can be different from those to the un-perturbed Riemann problem. In the un-perturbed Riemann problem, both links carry free flow with a flow-rate $q = d_1$, and capacity drop does not occur: a new state (point B' in the figure) propagates downstream along with a shock wave on the downstream link. However, if a small perturbation leads to an intermediate state U_0 (point A' in the figure) with $d_0 > s_2$, capacity drop occurs with a new state (C_1, C_*) on the upstream link (point A'' in the figure), a backward shock or rarefaction wave connecting U_0 to (C_1, C_*) initiates at $x = 0$, and a forward or backward shock wave connecting U_1 to U_0 initiates at $x = -L$. When the downstream wave connecting U_0 to (C_1, C_*) catches up the upstream one connecting U_1 to U_0 , a new shock wave connecting U_1 to (C_1, C_*) forms and propagates upstream. On the downstream link, a new state

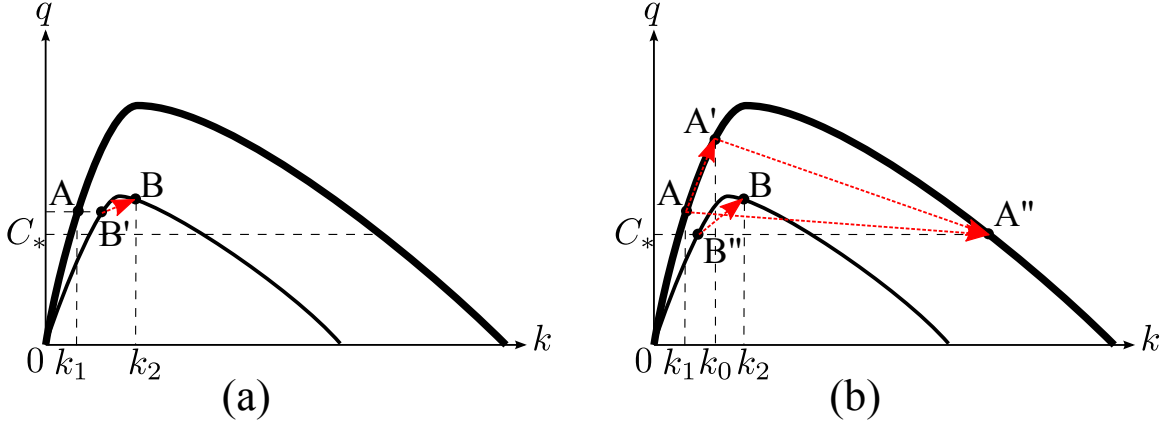


Figure 4: Kinematic wave solutions of (1) with an initial upstream condition at A and an initial downstream condition at B in the capacity-drop model: (a) without perturbations, (b) with perturbations

(point B'' in the figure) propagates downstream along with a shock wave. In this case, a sufficiently large perturbation to the initial condition can result in totally different solutions. However, if the perturbation is too small such that $d_0 \leq s_2$, capacity drop still does not occur. Therefore, the LWR model (1) with entropy condition (5) is bistable in this case.

When the road with a lane drop in Figure 1 carries free flow with a flow-rate greater than C_* , traffic breakdown and capacity drop can also be induced by fluctuations in both upstream demand and downstream supply. We demonstrate the process in Figure 5. If initially the upstream link carries a uniform, free flow traffic at (d_1, C_1) (point A in Figure 5) and the downstream link carries a uniform, free flow traffic at (d_1, C_2) (point B in Figure 5), where $C_* < d_1 \leq C_2$.

1. If a platoon of vehicles from the upstream link, which has a high density with a demand greater than C_2 (point A' on Figure 5(a)), reaches the lane-drop bottleneck, then vehicles queue up on the upstream link, capacity drop is activated, and traffic on the upstream link breaks down and becomes (C_1, C_*) (point A'' in Figure 5(a)). Correspondingly, traffic on the downstream link becomes (C_*, C_2) (point B' in Figure 5(a)). The throughput drops from d_1 to C_* .
2. If a downstream congested queue, which has a supply smaller than d_1 (point B' in Figure 5(b)), propagates to the lane-drop area, then vehicles queue up on the upstream link, capacity drop is activated, and traffic on the upstream link breaks down and becomes (C_1, C_*) (point A' in Figure 5(b)). Correspondingly, traffic on the downstream link becomes (C_*, C_2) (point B'' in Figure 5(b)). The throughput drops from d_1 to C_* .

In both cases, even if the fluctuations are instantaneous but sufficiently large, the induced capacity drop will be sustained. This again confirms the bistability property of the traffic system.

The analyses of fluctuations in both initial and boundary conditions further confirm that the new kinematic wave model replicates the two main characteristics of capacity drop: (i) capacity

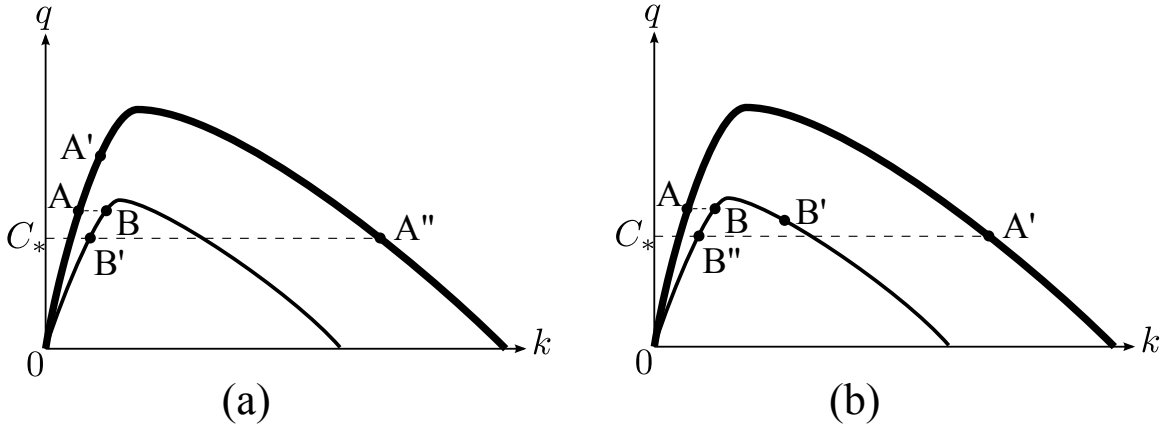


Figure 5: Activation of capacity drop: (a) A high-density platoon on the upstream link; (b) A congested queue on the downstream link

drop occurs with an upstream queue, and (ii) the discharge flow-rate drops once it is activated. In addition, as observed in real world (Persaud et al., 1998, 2001), the capacity drop as well as traffic breakdown can be induced by random fluctuations in upstream and downstream conditions even when the upstream is uncongested but carries a flow-rate higher than the dropped capacity, C_* .

Due to the bistability of the traffic system subject to fluctuations in both initial and boundary conditions, control methods should be sufficiently robust to avoid the activation of capacity drop by sudden increase in upstream demand or drop in downstream supply.

4 An empirical observation of capacity drop and flow-density relations at a bottleneck

In this section we present an empirical observation of the flow-density relation located at the merging section between I-405 South and Jeffrey Rd in Irvine, CA. Figure 6 shows the extended study site covering the freeway section between Jeffrey Rd and Sand Canyon Ave. Vehicle detector stations (VDS's) are installed at the freeway mainline and on-/off-ramps. In this study, we use data from the following VDS's: (i) VDS 1201171 and VDS 1201165 in the upstream location, (ii) VDS 1209189 in the middle location, and (iii) VDS 1201145 in the downstream location. The locations of these VDS's are also provided in Figure 6. Note that here the bottleneck location is between the upstream and middle detectors. Also note that, since we cannot find data for an active lane-drop bottleneck, we use a merge bottleneck to approximate a lane-drop bottleneck. This is a reasonable approximation, since the on-ramp lane can be considered as an additional mainline lane.

From PeMS (<http://pems.dot.ca.gov/>) we retrieve 30-second raw data from 5:00 AM to 10:00 PM in 34 weekdays from February to May in 2012, during which all detectors are healthy and have observation rates greater than 95%. Following (Cassidy, 1998), we identify near-stationary states at the three locations. By assuming a free-flow speed of 65 mph, we obtain the g-factors

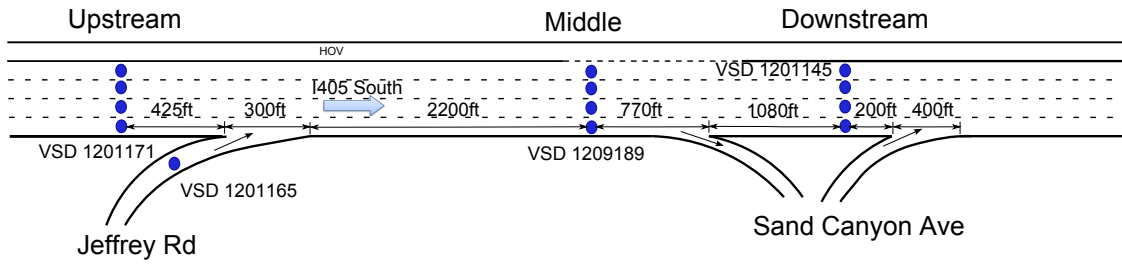


Figure 6: A section of the I-405 South freeway

as 21 ft for the upstream detector and 18 ft for the middle one and then convert occupancies into densities using $k = occ * 5280 / g$, where k is the density and occ is the occupancy. Then we show the observed flow-density pairs in near-stationary states for upstream and middle detectors in Figure 7, where the circles are for the upstream location, and the asterisks for the middle location. In addition, we use different colors for traffic states in three different regimes: SUC-UC when both upstream and middle locations are uncongested (or free flow), SOC-SUC when the upstream location is congested, but the downstream not, and SOC-SOC when both upstream and middle locations are congested. In Figure 7, we also plot the approximate triangular traffic flow fundamental diagrams as well as two dots A and A'.

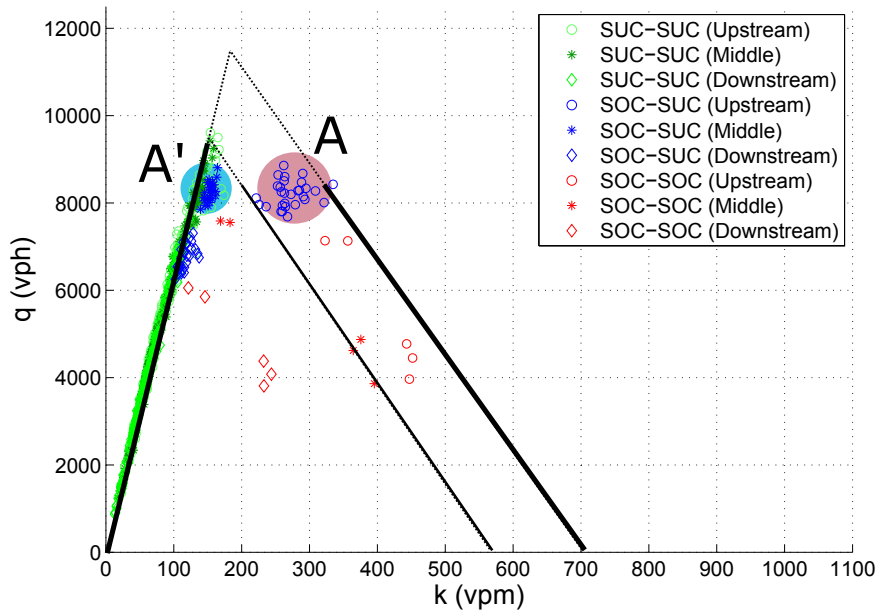


Figure 7: Flow-density relations in near-stationary states at the upstream and middle detectors

From Figure 7, we have the following observations consistent with those predicted in Figure

3(a): (i) There are three regimes of traffic conditions at the upstream and downstream location of a lane-drop or merging bottleneck, and the flow-rates are always equal. Note that the off-ramp flow-rates at Sand Canyon Ave are not available, and the flow-rates at the downstream location are consistently lower than those at the middle location. (ii) The maximum discharge flow-rate is about 9500 vph when all three locations are uncongested. (iii) When the bottleneck is activated, the upstream is congested at A and the downstream uncongested at A', and the average discharge flow-rate is about 8500 vph, with a capacity drop magnitude of 10.5%. In this case traffic at the downstream location is also uncongested, as shown in the figure. (iv) Some traffic states cannot be observed at these three locations, and the flow-density relation is discontinuous. This further confirms the validity of the assumptions and conclusions of the new capacity drop model.

5 Conclusion

In this paper, we proposed a phenomenological model of capacity drop within the framework of kinematic wave theories. Here the fundamental diagrams are still continuous, and, thus, the new model is devoid of the fallacy of models based on discontinuous fundamental diagrams. But, for capacity drop occurring at a lane drop location, we introduced a new entropy condition, in which the boundary flux is discontinuous in the upstream demand and downstream supply and reduced to a dropped capacity when the upstream demand is higher than the downstream supply. We showed that the model is well-defined for the Riemann problem. We further demonstrated that the new model replicates the three characteristics of capacity drop: (i) the maximum discharge flow-rate equals the downstream capacity in the SUC-UC regime; (ii) the capacity drops when the bottleneck is activated; i.e., the discharge flow-rate drops in the SOC-SUC regime; and (iii) some steady states are unobservable at the upstream and downstream locations, and the observed flow-density relations appear to be discontinuous at these locations. The three characteristics are further verified by empirical observations at a merge bottleneck, which is used to approximate a lane-drop bottleneck. These theoretical and empirical results verify that the shape of a fundamental diagram depends on the location of observations relative to the bottleneck (Hall and Agyemang-Duah, 1991). For the purpose of comparison, we compared the LWR model with and without capacity drop and concluded that the new flux function can replicate the capacity drop phenomenon.

In this study, we showed that the traffic system is bistable with capacity drop subject to perturbations in initial and boundary conditions, since traffic breakdown and capacity drop can be triggered by sufficiently large perturbations in initial and boundary conditions. This is a very important insight for analysis and design of a variable speed limit control system at a freeway lane-drop bottleneck (Jin and Jin, 2014a,b). In the future, we will be interested in extending the new model of capacity drop for general merge bottlenecks and evaluating and developing traffic control strategies, including variable speed limits and ramp metering, to delay or avoid the occurrence of capacity drop.

Many existing junction models in the literature can be extended by introducing fixed capacity constraints. For example, one can revise (4) by introducing a reduced capacity as,

$$q(x,t) = \min\{d(x^-,t),s(x^+,t),\bar{C}\},$$

where \bar{C} is smaller than both upstream and downstream capacities and can be caused by various stationary or moving bottlenecks. Note that, however, this model is different from the capacity drop model in (5): (i) the junction flux is still continuous in the upstream demand and downstream supply by following arguments in the proof of Theorem 2.1; and (ii) this model cannot capture the first or second characteristics of capacity drop. Therefore, fixed capacity constraints cannot describe the capacity drop phenomenon. In addition, mathematically speaking, this new model is fundamentally different from existing ones, which have continuous junction flux functions, as it is well known that differential equations with discontinuous right-hand sides are fundamentally different from those with continuous right-hand sides. Furthermore, both location- and time-dependent capacity constraints have been incorporated into the Hamilton-Jacobi equation of the LWR model in (Daganzo, 2005; Mazaré et al., 2011), either through the variational principle or the Hopf-Lax formula. But demand and supply were not defined in these studies, and location- and time-dependent capacity constraints cannot replicate the three characteristics of the capacity drop phenomenon, which contains state-dependent capacity constraints. That said, however, the capacity drop model in (5) can be incorporated into the Link Transmission Model, in which demand and supply are defined based on the Hopf-Lax formula (Jin, 2015).

The new model is phenomenological and lacks microscopic mechanism related to driving behaviors at an active bottleneck, since (i) the magnitude of Δ is exogenous and has to be calibrated for each study site; (ii) capacity drop occurs at one point, as shown in Figure 1, but a transition region of 1-2km long can usually be observed around an active bottleneck with capacity drop (Cassidy and Bertini, 1999; Cassidy and Rudjanakanoknad, 2005); and (iii) capacity drop occurs immediately after the upstream is congested, but in reality only after a number of vehicles queue up on the shoulder lane and lane changes disrupt traffic on all lanes (Cassidy and Rudjanakanoknad, 2005). In the literature, there have been many studies on the behavioral mechanism of capacity drop. It was observed that, when an active bottleneck stabilizes, there is an acceleration zone around the bottleneck (Banks, 1991b), and it was conjectured that the reduced flow is a consequence of the way drivers accelerate away from the queue (Hall and Agyemang-Duah, 1991; Papageorgiou et al., 2008). In (Cassidy and Rudjanakanoknad, 2005), it was observed that the occurrence of capacity drop at a merging bottleneck is associated with an extensive queue on the shoulder lane upstream to the merging point, sharp declines in vehicle speeds, and increases in lane-changing activities. However, it was pointed out that lane changing alone might not explain the capacity drop. Even though there have been many studies on capacity drop caused by heterogeneous drivers (Daganzo, 2002; Chung and Cassidy, 2004), pedestrians (Jiang et al., 2002), buses (Zhao et al., 2007), or accidents (Knoop et al., 2008), the causes and mechanism of capacity drop at active bottlenecks remain to be clarified. In (Persaud et al., 1998, 2001), traffic breakdown and capacity drop were found to be related to the upstream traffic demand randomly. The occurrence and magnitude of capacity drop have been successfully replicated in microscopic or hybrid simulations (Tampere et al., 2005; Treiber et al., 2006; Laval and Daganzo, 2006; Carlson et al., 2010; Leclercq et al., 2011). In the future, we will be interested in finding the relationship between the capacity drop magnitude and the road geometry, vehicles' acceleration rates, and traffic conditions and incorporating it the kinematic wave model to analyze traffic dynamics inside the transition region during the transition period. Such a behavioral model can be used to further develop control and design strategies for

critical road bottlenecks.

Acknowledgments

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Appendix A. Proof of Theorem 2.2

Proof. From the feasibility conditions on stationary and interior states, we can see that $q \leq d_1$ and $q \leq s_2$. Therefore, $q \leq \min\{d_1, s_2\}$. We first solve the flow-rate in the following four cases.

1. When $d_1 \leq \min\{s_2, C_*\}$, we assume that $q < d_1$. Thus we have $U_1^* = U_1^0 = (C_1, q)$ and $U_2^* = U_2^0 = (q, C_2)$. Thus $d_1^0 = C_1 > s_2^0 = C_2$. However, from (9) we have that $q = C_*$, which contradicts $q < d_1 \leq C_*$. Thus in this case $q = d_1$.
2. When $C_* < d_1 \leq s_2 \leq C_2 < C_1$, we consider the following three scenarios:
 - First, if $q = d_1 \leq s_2 < C_1$, we have $U_1^* = U_1^0 = (q, C_1)$. If $d_1 < s_2$, then $U_2^* = U_2^0 = (q, C_2)$; if $d_1 = s_2$, then $U_2^* = (C_2, q)$, and U_2^0 is between (C_2, q) and (q, C_2) . In this case $d_1^0 \leq s_2^0$, which satisfy (9). Thus $q = d_1$, $U_1^* = (q, C_1)$, and $U_2^* = (q, C_2)$ ($d_1 < s_2$) or $U_2^* = (C_2, q)$ ($d_1 = s_2$) satisfy (9).
 - Second, if $q < d_1 \leq s_2$ and $q \neq C_*$, we have $U_1^* = U_1^0 = (C_1, q)$ and $U_2^* = U_2^0 = (q, C_2)$, which lead to $d_1^0 = C_1 > s_2^0 = C_2$. However from (9) we have $q = C_*$, which contradicts $q \neq C_*$. Thus it is impossible to have that $q < d_1$ and $q \neq C_*$.
 - Third, if $q = C_* < d_1 \leq s_2$, we have $U_1^* = U_1^0 = (C_1, q)$ and $U_2^* = U_2^0 = (q, C_2)$. Thus $d_1^0 = C_1 > s_2^0 = C_2$, which satisfies (9). Thus $q = C_*$, $U_1^* = (q, C_1)$ and $U_2^* = (q, C_2)$ satisfy (9).

Therefore, both $q = d_1$ and $q = C_*$ satisfy (9). However, from (8), the unique solution of the boundary flux is $q = d_1 > C_*$.

3. When $d_1 > s_2$ and $s_2 \leq C_*$, if $q < s_2$, then $U_1^* = U_1^0 = (C_1, q)$ and $U_2^* = U_2^0 = (q, C_2)$, which lead to $d_1^0 = C_1 > s_2^0 = C_2$. However from (9) we have $q = C_*$, which contradicts $q < s_2 \leq C_*$. Thus $q = s_2$.
4. When $d_1 > s_2 > C_*$, we consider the following three scenarios:
 - First, if $q > C_*$ and $q < s_2 < d_1$. Then $U_1^* = U_1^0 = (C_1, q)$, and $U_2^* = U_2^0 = (q, C_2)$, which lead to $d_1^0 = C_1 > s_2^0 = C_2$. However from (9) we have $q = C_*$, which contradicts $q > C_*$.

- Second, if $q > C_*$ and $q = s_2 < d_1$. Then $U_1^* = U_1^0 = (C_1, q)$, $U_2^* = (C_2, q)$, and $d_2^0 > q$. Since $d_1^0 = C_1 > C_2 \geq s_2^0$, from (9) we have $q = \min\{s_2^0, C_*\} \leq C_*$, which contradicts $q > C_*$.
- Third, if $q < C_* < s_2 < d_1$. Then $U_1^* = U_1^0 = (C_1, q)$, and $U_2^* = U_2^0 = (q, C_2)$, which lead to $d_1^0 = C_1 > s_2^0 = C_2$. However from (9) we have $q = C_*$, which contradicts $q < C_*$.

Therefore, $q = C_*$.

In all of the four cases, the boundary flux is uniquely solved by

$$q = \begin{cases} d_1, & d_1 \leq s_2 \\ \min\{s_2, C_*\}, & d_1 > s_2 \end{cases}$$

Note that (9) cannot be used to pick out a unique solution in q when $C_* < d_1 \leq s_2$. Therefore, (9) is a necessary condition, but not sufficient. In contrast, (8) is both necessary and sufficient.

From the feasibility conditions on the stationary states, $U_1^* = (d_1, C_1)$ when $q = d_1$, and $U_1^* = (C_1, q)$ otherwise. Similarly, $U_2^* = (C_2, s_2)$ when $q = s_2$, and $U_2^* = (q, C_2)$ otherwise.⁶ That is, the stationary states are uniquely solved. With the stationary states, we can solve the traditional LWR model to find shock or rarefaction waves on each link. ■

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⁶Note that interior states may not be uniquely solved, but they do not impact the kinematic wave solutions.

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